Effect of resistivity on the Rayleigh-Taylor instability in an accelerated plasma

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(Received 17 May 1993)

We study the Rayleigh-Taylor instability in finite-conductivity accelerated plasma arcs of the type found in electromagnetic rail launchers. For a plasma of length l, acceleration a, and thermal speed v_T we consider the case where $v_T^2/al \gg 1$, which is valid when the projectile mass is large compared to the plasma mass. The conductivity σ enters via a magnetic Reynolds number $R=\sigma\mu(al^3)^{1/2}$. The fourth-order mode equation is solved analytically using an asymptotic WKB expansion in $1/R$. We find the first-order $1/R$ correction to the classical Rayleigh-Taylor dispersion relation for large wave number K but with $K \ll R^2/l$. The analytical results show good agreement with previous numerical calculations.

PACS number(s): 52.35.Py, 52.75.Di, 52.30.Jb, 47.20.−k

I. INTRODUCTION

It is well known that the state where a dense fluid rests above a light fluid, with the weight of the dense fluid supported by the pressure of the light fluid, is unstable. The instability results in the growth of waves on the interface that ultimately lead to the interchange of the fluid positions. This Rayleigh-Taylor [1,2] (RT) instability was first studied for the case of incompressible nonconducting fluids. Kruskal and Schwarzschild [3] treated the similar magnetohydrodynamics (MHD) problem of an incompressible infinitely conducting horizontal plasma slab, with a sheet surface current at its bottom surface, sitting above a uniform horizontal magnetic field. Here the weight of the plasma is supported by the magnetic force on the sheet current. For a gravitational field g in the negative X direction and a magnetic field in the Z direction, a normal model of the (RT) instability has the form

$$e^{iKY}e^{\Omega t} .$$

For the plasma case the disturbance of the surface has the shape of flutes along the magnetic field direction (if $K_y=0$, $K_z\neq 0$ the modes describe the filamentation instability). Both for the fluid and plasma cases the RT instability has the well-known growth rate

$$\Omega_{\mathrm{RT}}=\sqrt{Kg} . \tag{1}$$

The same RT (also called fluting, interchange, or Kruskal-Schwarzschild) instability plays a role in many situations in which a plasma is accelerated by $\mathbf{J}\times\mathbf{B}$ forces because a pseudogravitational force appears in the frame of reference of the accelerated plasma. Under these circumstances, the effect of finite electrical conductivity (nonzero resistivity) upon the growth of the MHD interchange instability was first treated by Tsai, Liskow, and Wilcox [4] in the context of plasma implosions for intense x-ray generation. However, they included the effects of finite conductivity only so far as allowing a magnetic-field profile in the initial plasma equilibrium, rather than zero field with a sheet current as in Ref. [3]. They neglected the effect of the conductivity upon the linearized time-development perturbation equations which led to a second-order mode equation making the problem quite similar to that of infinite conductivity. Powell [5] used the same approximation of Ref. [4] to discuss the RT instability in the accelerated plasma arcs produced in electromagnetic rail launchers. Decker, Huerta, and Rodriguez-Trelles [6] went beyond this and included in their numerical work the effects of finite conductivity upon the perturbation equations, which led to a complicated fourth-order mode equation. Bourouis, Huerta, and Rodriguez-Trelles [7] carried the problem further but also treated the problem numerically.

The main thrust of this paper is to obtain analytical results for the effect of nonzero resistivity (finite conductivity) in accelerated plasma arcs of the type found in electromagnetic rail launchers. (Fig. 1 is a schematic of the device.) The current flows into a conducting rail, across a hot, dense plasma arc, and returns via the other rail. The arc experiences strong $\mathbf{J}\times\mathbf{B}$ forces that compress it against the rear of the projectile. In Sec. II we write down the equations that govern the density, pressure, velocity, and magnetic field in the plasma. We use a coordinate frame accelerating with the projectile, introducing in this way a pseudogravitational force. In the accelerated frame the arc has a steady state of length l as described in Sec. III. Typically the projectile mass is much greater than the plasma mass so the acceleration a is much lower than it would be in a free-running arc. When the appropriate length and time scales are used to make the problem dimensionless, we find that the equations contain two dimensionless parameters. The first one is

$$\Lambda=\frac{v_T^2}{al} , \tag{2}$$

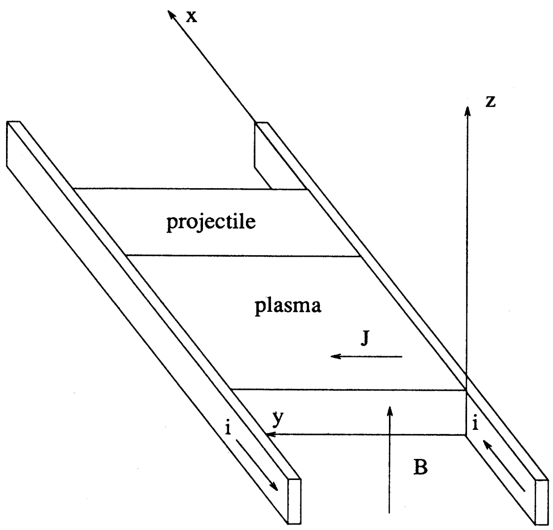

FIG. 1. Schematic description of the barrel of an idealized rail launcher. The plasma behind the projectile is accelerated by the $\mathbf{J}\times\mathbf{B}$ forces.

where v_T is the thermal speed. In Sec. IV we show that when Λ is large the pressure and $\mathbf{J}\times\mathbf{B}$ forces dominate the acceleration terms in the force equation, but they balance each other in a kind of time-dependent equilibrium. The second parameter is a magnetic Reynolds number

$$R=\sigma\mu(al^3)^{1/2} , \tag{3}$$

where μ is the magnetic permeability and σ is the electrical conductivity. The linear stability analysis is done in Sec. V. In the limit $\Lambda\gg 1$ we are able to do a WKB asymptotic expansion using $1/R$ as the expansion parameter. We show that the solution can be separated into two different parts; a part $\hat{A}$ that is algebraic in R, plus a part $\hat{E}$ that is exponentially small in R. To leading order in the expansion, the two parts may be studied separately. The leading-order contribution (in the $1/R$ expansion) to the algebraic part is studied in Sec. VII. Here an asymptotic analysis is carried out for large disturbance wave numbers k by dividing the interval of interest into four regions. The matching between the four regions leads to the dispersion relation of Eq. (103), which is one of the main results of the paper. This dispersion relation is simplified in both the small- and large-wave-number limits, leading to Eqs. (109) and (112), respectively. Our results are compared with the results of Ref. [5] which were obtained by a direct numerical integration of the equations. The first correction in the $1/R$ expansion to the dispersion relation is obtained in Sec. VIII. This shows that the electrical resistivity plays a stabilizing role, as expected, and moreover, that the stabilizing effects increase linearly with the disturbance wave number. Equation (125) is a simplified asymptotic dispersion relation for large wave-number disturbances and large magnetic Reynolds numbers. It summarizes the main results of the present analysis. In Sec. IX the range of validity of our results is indicated. Section X is devoted to a discussion of the results. The analysis of the exponentially small contribution $\hat{E}$ is done in the Appendix.

II. GOVERNING EQUATIONS

Our physical model for the plasma is the same as that of Ref. [5] as appropriate for rail launchers. We consider a plasma composed of electrons, and ions of a single species of mass m_0 in various stages of ionization. The plasma is assumed to be in a collision-dominated regime where it can be treated by the equations of MHD for a single conducting fluid. As in Ref. [5] a number of simplifying assumptions are made to define a quasi-two-dimensional problem that allows analytical treatment. We choose the X axis along the direction of the plasma acceleration. The confining rails are taken as perfectly conducting infinite planes parallel to the X-Z plane. The current flows through the plasma mostly in the Y direction and produces a magnetic field in the Z direction. We consider a two-dimensional model where all quantities vary only as function of X and Y but are independent of Z. For convenience, the coordinate system moves with the accelerated projectile and all quantities are measured in the accelerated frame of reference. The mechanical properties of the plasma are the density ρ, the pressure p, and the velocity $\mathbf{U}$. The electrical quantities are the current density $\mathbf{J}$, the magnetic field $\mathbf{B}$, and the electric field $\mathbf{E}$.

The MHD equations describing the mechanical behavior of the plasma arc are the conservation of mass

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\mathbf{U})=0 , \tag{4}$$

and the force equation

$$\rho\frac{\partial\mathbf{U}}{\partial t}+\rho\mathbf{U}\cdot\nabla\mathbf{U}=-\nabla p+\mathbf{J}\times\mathbf{B}-\rho a\hat{\mathbf{x}} , \tag{5}$$

where a, which we take constant, is the acceleration of the projectile and of our noninertial reference frame. The choice of this noninertial frame introduces the pseudoforce $-\rho a\hat{\mathbf{x}}$, where $\hat{\mathbf{x}}$ is the unit vector in the X direction. The temperature T is considered constant as in Refs. [4–7] and the equation of state is taken as

$$p=v_T^2\rho , \tag{6}$$

where the thermal speed v_T, taken as a constant, is given by

$$v_T^2=\frac{(1+\beta)k_BT}{m_0} . \tag{7}$$

Here β is a function of T such that $\rho(1+\beta)/m_0$ is the total particle density (neutrals plus ions plus electrons), and k_B is Boltzmann's constant.

The magnetic properties are described by

$$\frac{\partial\mathbf{B}}{\partial t}=-\nabla\times\mathbf{E} , \tag{8}$$

$$\nabla\cdot\mathbf{B}=0 , \tag{9}$$

$$\nabla\times\mathbf{B}=\mu\mathbf{J} , \tag{10}$$

and

$$\mathbf{J}=\sigma(\mathbf{E}+\mathbf{U}\times\mathbf{B}) , \tag{11}$$

where the electrical conductivity σ is taken as a constant. As usual we eliminate the electric field using Eq. (11) and the current density using Eq. (10), rewriting Eq. (5) as

$$\rho\frac{\partial \mathbf{U}}{\partial t}+\rho\mathbf{U}\cdot\nabla\mathbf{U}=-v_T^2\nabla\rho+\frac{1}{\mu}(\nabla\times\mathbf{B})\times\mathbf{B}-\rho a\hat{\mathbf{x}} , \tag{12}$$

and Eq. (8) as

$$\frac{\partial \mathbf{B}}{\partial t}=\nabla\times(\mathbf{U}\times\mathbf{B})+\frac{1}{\sigma\mu}\nabla^2\mathbf{B} . \tag{13}$$

In the vacuum region behind the plasma

$$\nabla\times\mathbf{B}^v=0 , \tag{14}$$

where the superscript v denotes quantities in the vacuum.

Our two-dimensional geometry imposes that all quantities $\rho(X,Y,t)$, $p(X,Y,t)$, $U_x(X,Y,t)$, $U_y(X,Y,t)$ and $B_z(X,Y,t)$ are independent of Z, so

$$\frac{\partial}{\partial z}=0 \quad\text{and}\quad U_z(X,Y,t)=B_x(X,Y,t)=B_y(X,Y,t)=0 . \tag{15}$$

In order to specify completely the problem we have to establish the appropriate boundary conditions at the plasma-projectile solid surface, where $X=l$, at the plasma-vacuum interface, where $X=S(Y,t)$, and at the rail surfaces, located at $Y=0$ and w. The projectile is solid and carries no current. Therefore

$$U_x(l,Y,t)=0 \quad\text{and}\quad \mathbf{B}(l,Y,t)=\mathbf{0} . \tag{16}$$

On the other hand, the plasma-vacuum interface is located at $X=S(Y,t)$ and it moves with the local plasma velocity. This kinematic condition provides the equation

$$\frac{\partial S}{\partial t}=U_x-\frac{\partial S}{\partial Y}U_y . \tag{17}$$

Since the plasma is not a perfect conductor there is no sheet current at the plasma-vacuum interface, therefore

$$\mathbf{B}(S(Y,t),Y,t)=\mathbf{B}^v \quad\text{and}\quad p(S(Y,t),Y,t)=0 . \tag{18}$$

At the rails we have that the normal component of the fluid velocity is zero,

$$U_y(X,0,t)=U_y(X,w,t)=0 . \tag{19}$$

At the infinitely conducting rails we have the tangential component of the electric field E_x as zero. Using Eqs. (11) and (10) this translates into zero for the normal derivative of the tangential magnetic field

$$\frac{\partial B_z(X,0,t)}{\partial y}=\frac{\partial B_z(X,w,t)}{\partial y}=0 . \tag{20}$$

III. STEADY STATE

We denote by the subscript 0 the quantities in the steady state. The plasma-vacuum interface is at the location $X_0=S_0(Y,t)=0$. The plasma is at rest, $\mathbf{U}_0=\mathbf{0}$, in the accelerated frame that moves with the projectile. That is, the plasma moves with a constant depth l as it follows behind the accelerated projectile. The steady-state magnetic field is given by

$$B_{0,Z}(X,Y)=\mu j(1-X/l) \quad\text{for } 0\le X\le l ,$$
$$B_{0,Z}^v(X,Y)=\mu j \quad\text{for } X\le 0 , \tag{21}$$

where $j=lJ_{0,Y}$ is the steady-state current per unit rail height. The steady-state density is given by

$$\rho_0=\frac{\mu j^2}{al}\left\{(1+\Lambda)[1-e^{-X/(l\Lambda)}]-\frac{X}{l}\right\} , \tag{22}$$

where Λ was defined in Eq. (2). This parameter Λ can be expressed in terms of commonly used dimensionless parameters as the Froude number divided by the square of the plasma Mach number.

IV. DIMENSIONLESS EQUATIONS

We introduce dimensionless quantities by taking l as the length scale and $(l/a)^{1/2}$ as the time scale. We define

$$x,y,z=\frac{X}{l},\frac{Y}{l},\frac{Z}{l} \quad\text{and}\quad \tau=t\left[\frac{a}{l}\right]^{1/2} , \tag{23}$$

as the dimensionless space and time coordinates. We define dimensionless density, velocity, and magnetic-field variables

$$r=\frac{v_T^2}{\mu j^2}\rho , \quad \mathbf{u}=\frac{\mathbf{U}}{(la)^{1/2}} , \quad \mathbf{b}=\frac{\mathbf{B}}{\mu j} . \tag{24}$$

Referring to these new variables, the steady state described in the preceding section corresponds to

$$\mathbf{u}_0=\mathbf{0} , \quad r_0=\Lambda[(1+\Lambda)(1-e^{-x/\Lambda})-x] ,$$
$$b_{0,z}=1-x . \tag{25}$$

The time development equations (4), (12), and (13) become

$$\frac{\partial r}{\partial \tau}+\nabla\cdot(r\mathbf{u})=0 , \tag{26}$$

$$r\frac{\partial \mathbf{u}}{\partial \tau}+r\mathbf{u}\cdot\nabla\mathbf{u}=-\Lambda\nabla r+\Lambda(\nabla\times\mathbf{b})\times\mathbf{b}-r\hat{\mathbf{x}} , \tag{27}$$

where it is clear that the pressure and magnetic forces dominate and balance in the $\Lambda\gg 1$ limit, and

$$\frac{\partial \mathbf{b}}{\partial \tau}=\nabla\times(\mathbf{u}\times\mathbf{b})+\frac{1}{R}\nabla^2\mathbf{b} , \tag{28}$$

where R is a magnetic Reynolds number defined in Eq. (3). Here the conductivity enters as a magnetic diffusivity D_m, or viscosity ν_m,

$$D_m=\frac{1}{\mu\sigma} . \tag{29}$$

R accounts for the relative importance of magnetic diffusion (or magnetic viscosity) effects. A large value of R corresponds to small D_m in a good conductor, where the magnetic disturbances are convected by the plasma volume elements along their pathlines, with small diffusion to adjacent volume elements. A small value of

R corresponds to a high value for D_m which means that a magnetic perturbation will diffuse rapidly through a low conductivity plasma.

The boundary conditions of Eq. (16) at the plasma-projectile surface become

$$u_x=0 \quad \text{and} \quad \mathbf{b}=0 \quad \text{at } x=1 \ . \tag{30}$$

At the plasma-vacuum interface, the boundary conditions of Eq. (18) become

$$\mathbf{b}=\mathbf{b}^v \quad \text{and} \quad r=0 \quad \text{at } x=s(y,\tau) \ , \quad \text{where } s=S/l \ . \tag{31}$$

The time evolution of s, from Eq. (17), becomes

$$\frac{\partial s}{\partial \tau}=u_x-\frac{\partial s}{\partial y}u_y \quad \text{at } x=s(y,\tau) \ . \tag{32}$$

V. LINEAR STABILITY ANALYSIS

We study the growth of fluctuations that disrupt the equilibrium of Eq. (25). We use the standard method of introducing a small perturbation of the stationary state and doing a linear mode analysis. We seek linearized solutions of the form

$$F(x,y,z,\tau)=F_0(x)+F_1(x,y,\tau) \ , \tag{33}$$

where F_0 is the stationary solution and $F_1(x,y,\tau)$ is a small perturbation. We linearize the time evolution equations (26)–(28) around the stationary state

$$\frac{\partial r_1}{\partial \tau}+\frac{\partial}{\partial x}(r_0u_{1,x})+\frac{\partial}{\partial y}(r_0u_{1,y})=0 \ , \tag{34}$$

$$r_0\frac{\partial u_{1,x}}{\partial \tau}=-\Lambda\frac{\partial r_1}{\partial x}-\Lambda\frac{\partial}{\partial x}(b_{0,z}b_{1,z})-r_1 \ , \tag{35}$$

$$r_0\frac{\partial u_{1,y}}{\partial \tau}=-\Lambda\frac{\partial r_1}{\partial y}-\Lambda b_{0,z}\frac{\partial b_{1,z}}{\partial y} \ , \tag{36}$$

$$\frac{\partial b_{1,z}}{\partial \tau}=-\frac{\partial}{\partial x}(b_{0,z}u_{1,x})-\frac{\partial}{\partial y}(b_{0,z}u_{1,y}) +\frac{1}{R}\left[\frac{\partial^2 b_{1,z}}{\partial x^2}+\frac{\partial^2 b_{1,z}}{\partial y^2}\right] . \tag{37}$$

The boundary conditions of Eq. (30) become

$$u_{1,x}=b_{1,z}=0 \quad \text{at } x=1 \ . \tag{38}$$

The plasma-vacuum interface is located at $x=s(y,\tau)$. Since s is small we can refer the boundary conditions to the line $x=0$ by a simple Taylor expansion. Then, keeping only the linear terms in the perturbations in Eq. (31), we obtain

$$r_1+\frac{dr_0}{dx}s=0 \quad \text{and} \quad b_{1,z}+s\frac{db_{0,z}}{dx}=0 \quad \text{at } x=0 \ , \tag{39}$$

while, from Eq. (32)

$$\frac{\partial s}{\partial \tau}=u_{1,x} \quad \text{at } x=0 \ . \tag{40}$$

The system of Eqs. (34)–(37) can be solved by expanding each variable in a Fourier series in y. Since the system is linear in the disturbances, we can study the behavior of the stationary state via each Fourier mode. Taking into account the boundary conditions of Eqs. (19) and (20) at the rails we write

$$r_1(x,y,\tau)=\hat{r}(x)\cos(ky)e^{\Gamma\tau} \ , \tag{41}$$

$$u_{1,x}(x,y,\tau)=\hat{u}(x)\cos(ky)e^{\Gamma\tau} \ , \tag{42}$$

$$u_{1,y}(x,y,\tau)=\hat{v}(x)\sin(ky)e^{\Gamma\tau} \ , \tag{43}$$

$$b_{1,z}(x,y,\tau)=\frac{\hat{b}(x)}{\Gamma}\cos(ky)e^{\Gamma\tau} \ , \tag{44}$$

and

$$s(y,\tau)=\hat{s}\cos(ky)e^{\Gamma\tau} \ , \tag{45}$$

where for convenience we have introduced the factor Γ in the definition of $\hat{b}(x)$. Γ is the dimensionless growth rate defined in terms of the growth rate Ω as

$$\Gamma=\Omega\sqrt{l/a} \ , \tag{46}$$

and k is the dimensionless wave number of the perturbation defined in terms of the wave number K as

$$k=Kl \ . \tag{47}$$

The choice of sines and cosines is ruled by the rail boundary conditions and restricts the wave number to the values

$$k=n\pi\frac{l}{w} \ , \quad \text{where } n=1,2,\ldots \ . \tag{48}$$

The results in what follows would be essentially the same if we had taken all the perturbations proportional to e^{iky}, as would be appropriate for an infinite rail separation. Thus we will treat k as a continuous variable.

Substituting Eqs. (41)–(45) into Eqs. (34)–(37) we find that the x dependence of the disturbances is obtained from the equations

$$\Gamma\hat{r}+\frac{d}{dx}(r_0\hat{u})+kr_0\hat{v}=0 \ , \tag{49}$$

$$\Gamma r_0\hat{u}+\Lambda\frac{d\hat{r}}{dx}+\frac{\Lambda}{\Gamma}\frac{d}{dx}(b_{0,z}\hat{b})+\hat{r}=0 \ , \tag{50}$$

$$\Gamma r_0\hat{v}-k\Lambda\hat{r}-k\frac{\Lambda}{\Gamma}b_{0,z}\hat{b}=0 \ , \tag{51}$$

and

$$\frac{1}{R\Gamma}\left[\frac{d^2\hat{b}}{dx^2}-k^2\hat{b}\right]-\hat{b}-\frac{d}{dx}(b_{0,z}\hat{u})-kb_{0,z}\hat{v}=0 \ . \tag{52}$$

Using Eq. (51) in Eq. (49), the variable $\hat{v}$ may be eliminated, with the result

$$(\Gamma^2+k^2\Lambda)\hat{r}+k^2\frac{\Lambda}{\Gamma}b_{0,z}\hat{b}+\Gamma\frac{d}{dx}(r_0\hat{u})=0 \ . \tag{53}$$

With Eqs. (51) and (53), $\hat{r}$ and $\hat{v}$ may be eliminated from Eqs. (50) and (52). The result is

$$\frac{d^2}{dx^2}(r_0\hat{u})+\frac{1}{\Lambda}\frac{d}{dx}(r_0\hat{u})-\left[k^2+\frac{\Gamma^2}{\Lambda}\right]r_0\hat{u} -\frac{d}{dx}(b_{0,z}\hat{b})+\frac{k^2}{\Gamma^2}b_{0,z}\hat{b}=0 \ , \tag{54}$$

and

$$\frac{1}{R\Gamma}\left[\frac{d^2\hat{b}}{dx^2}-k^2\hat{b}\right]-\left[1+\frac{k^2\Lambda}{\Gamma^2+k^2\Lambda}\frac{b_{0,z}^2}{r_0}\right]\hat{b} \\ -\frac{\Gamma^2}{\Gamma^2+k^2\Lambda}\frac{b_{0,z}}{r_0}\frac{d}{dx}(r_0\hat{u})-r_0\hat{u}\frac{d}{dx}\left[\frac{b_{0,z}}{r_0}\right]=0 . \tag{55}$$

The boundary conditions of Eq. (38) become for $\hat{u}$ and $\hat{b}$

$$\hat{u}=\hat{b}=0 \quad \text{at } x=1 , \tag{56}$$

whereas, at $x=0$, from Eq. (40) we obtain

$$\hat{s}=\hat{u}(x=0)/\Gamma , \tag{57}$$

which determines the deformation of the plasma-vacuum interface. Using Eqs. (53) and (57) together with the steady-state profiles of Eqs. (21) and (22) in the boundary conditions of Eq. (39), we obtain

$$\hat{b}=\hat{u} \quad \text{and } \hat{u} \text{ remains finite at } x=0 . \tag{58}$$

A nontrivial solution of Eqs. (54) and (55) with the boundary conditions of Eqs. (56) and (58) will provide the dispersion relation between the growth rate Γ and the wave number k. The steady state is unstable when $\Gamma>0$, and an initially small disturbance grows until the nonlinear terms (disregarded in the previous perturbation equations) become important. When $\Gamma<0$ the steady state is linearly stable against small disturbances.

The system of Eqs. (54) and (55) simplifies strongly in the limit $\Lambda \gg 1$, which seem to be a good approximation for practical systems because of the relatively low acceleration due to the mass of the projectile. This is not the case, however, in the acceleration of free arcs, where the acceleration is much greater. The Rashleigh and Marshall [8] rail launcher, for example, accelerated a 3-gm solid object to a speed of about 6 km/s in a distance of some 3 m. For their experiment Λ was approximately 20 and R was about 7. On the other hand, in the EMACK (Electromagnetic Accelerator) demonstration of Deis, Scherbath, and Ferrentino [9], with a much heavier projectile, the values were $\Lambda \approx 250$ and $R\approx 4$.

Thus in the following we will treat the limit of large Λ, which allows us to pursue the problem analytically. The limit $\Lambda \gg 1$ is not the same as the incompressible limit even though it is a high thermal speed limit. The effect of compressibility on the Rayleigh-Taylor instability in a nonconducting fluid was analyzed by Bernstein and Book [10] who showed that, in general, the growth rates are larger when the fluids are compressible than in the incompressible limit.

In the $\Lambda \gg 1$ limit Eq. (55) is rewritten as

$$r_0\hat{u}=-\frac{F(x)}{R\Gamma}\left[\frac{d^2\hat{b}}{dx^2}-k^2\hat{b}\right]+\hat{b}x\left[1-\frac{x}{2}\right] , \tag{59}$$

where we have introduced the function

$$F=\frac{[x(1-x/2)]^2}{(1-x)^2+x(1-x/2)} , \tag{60}$$

and have used that, in the $\Lambda \gg 1$ limit, the stationary state dimensionless density given by Eq. (25) may be approximated by

$$r_0=x(1-x/2) . \tag{61}$$

Using Eq. (59) in Eq. (54) we get the fourth-order differential equation for the x dependence of the magnetic-field disturbance,

$$\frac{1}{R\Gamma}\left\{\frac{d^4\hat{b}}{dx^4}F+2\frac{dF}{dx}\frac{d^3\hat{b}}{dx^3}+\frac{d^2\hat{b}}{dx^2}\left[\frac{d^2F}{dx^2}-2Fk^2\right]-2k^2\frac{dF}{dx}\frac{d\hat{b}}{dx}+\hat{b}\left[k^4F-k^2\frac{d^2F}{dx^2}\right]\right\} \\ -x\left[1-\frac{x}{2}\right]\frac{d^2\hat{b}}{dx^2}-(1-x)\frac{d\hat{b}}{dx}+\hat{b}\left[k^2x\left[1-\frac{x}{2}\right]-\frac{k^2}{\Gamma^2}(1-x)\right]=0 . \tag{62}$$

Using Eq. (59) in Eqs. (56) and (58) we get the boundary conditions for the magnetic disturbance

$$\hat{b}=\frac{d^2\hat{b}}{dx^2}=0 \quad \text{at } x=1 , \tag{63}$$

and

$$\hat{b} \text{ remains finite as } x\to 0 \text{ and } \lim_{x\to 0}\left[x\frac{d^2\hat{b}}{dx^2}\right]\to 0 . \tag{64}$$

VI. LARGE BUT FINITE VALUES OF R

Even in the limit of large values of Λ, Eq. (62) is too complicated for a complete analytical treatment. To get some insight into the problem, we analyze the problem in the limit of large R using the WKB method. We seek an asymptotic solution of Eq. (62) of the form

$$\hat{b}=\left[\sum_{n=0}^{\infty}\hat{b}^{[n]}(x)R^{-n/2}\right]\exp[R^{1/2}q(x)] , \tag{65}$$

with a similar expansion for the amplification factor $\Gamma(k,R)$,

$$\Gamma=\sum_{n=0}^{\infty}\Gamma^{[n]}R^{-n/2} , \tag{66}$$

with the $\Gamma^{[n]}$ being functions of the wave number k only.

Substituting the expansions of Eq. (65) into Eq. (62) we find that the solution to $O(R)$ requires

$$\left[\frac{dq}{dx}\right]^4 \frac{F}{\Gamma^{[0]}} - x\left[1-\frac{x}{2}\right]\left[\frac{dq}{dx}\right]^2 = 0 \; . \tag{67}$$

Equation (67) has four roots. Two of them correspond to $dq/dx=0$, which leads to functions $\hat{b}_1$ and $\hat{b}_2$ with algebraic expansions in R,

$$\hat{b}_1 = \sum_{n=0}^{\infty} \hat{b}_1^{[n]}(x) R^{-n/2} \; ,$$
$$\hat{b}_2 = \sum_{n=0}^{\infty} \hat{b}_2^{[n]}(x) R^{-n/2} \; . \tag{68}$$

The nonvanishing solutions of Eq. (67) are

$$q(x) = (\Gamma^{[0]})^{1/2} \int_0^x \left[\frac{(1-t)^2 + t(1-t/2)}{t(1-t/2)}\right]^{1/2} dt \; , \tag{69}$$

and $-q(x)$, which lead to expansions for the two other independent solutions $\hat{b}_3$ and $\hat{b}_4$ of Eq. (62),

$$\hat{b}_3 = \left[\sum_{n=0}^{\infty} \hat{b}_3^{[n]}(x) R^{-n/2}\right] \exp[R^{1/2} q(x)] \; ,$$
$$\hat{b}_4 = \left[\sum_{n=0}^{\infty} \hat{b}_4^{[n]}(x) R^{-n/2}\right] \exp[-R^{1/2} q(x)] \; , \tag{70}$$

which are exponential in R. Therefore the general solution for $\hat{b}$ will have the form

$$\hat{b} = c_1 \hat{b}_1 + c_2 \hat{b}_2 + c_3 \hat{b}_3 + c_4 \hat{b}_4 = \hat{A} + \hat{E} \; . \tag{71}$$

The coefficients c_i are determined by the boundary conditions. This may make the c_i dependent on R and affect the relative order of the algebraic and exponential contributions. The solution, as indicated by Eq. (71), has naturally been divided into a part that will turn out to be exponentially small in $R^{1/2}$,

$$\hat{E} = c_3 \hat{b}_3 + c_4 \hat{b}_4 = \sum_{n=0}^{\infty} \hat{E}^{[n]}(x) R^{-n/2} \tag{72}$$

(it is not apparent that the $c_3 \hat{b}_3$ term will be exponentially small but c_3 will make it so), and a part that is algebraic in R,

$$\hat{A} = c_1 \hat{b}_1 + c_2 \hat{b}_2 = \sum_{n=0}^{\infty} \hat{A}^{[n]}(x) R^{-n/2} \; . \tag{73}$$

From Eq. (63) the magnetic-field perturbation of Eq. (71), and its second derivative, have to vanish at $x=1$. The vanishing of the second derivative of $\hat{b}$ at $x=1$, to leading order in both the algebraic and the exponential contributions, leads to the relation

$$\left[\frac{d^2 \hat{A}}{dx^2}\right]^{[\mathrm{LO}]} + R\,\Gamma^{[0]} \hat{E}^{[0]} = 0 \quad \text{at } x=1 \; , \tag{74}$$

where [LO] stands for the nonvanishing term of leading order in the $R^{-1/2}$ expansions. If we choose the leading order of $\hat{A}$ to be of order unity, then from Eq. (68), $(d^2\hat{A}/dx^2)^{[\mathrm{LO}]}$ will also be of order unity, at most. Therefore Eq. (74) shows that $E^{[0]}$ (which is multiplied by R) should be of smaller order that the contribution $\hat{A}^{[0]}$. Then, to leading order, $E^{[0]}$ will not appear in the boundary condition of Eq. (63) which imposes the vanishing of $\hat{b}$ at $x=1$, because there it will not be multiplied by the factor R and will be negligible compared to the term $\hat{A}^{[0]}$. [In fact, as we will see in the Appendix, the nonvanishing leading order of $d^2\hat{A}/dx^2$ (at $x=1$) is equal to $R^{-1} d^2 \hat{A}^{[2]}/dx^2$ (at $x=1$) because the second derivatives of both $\hat{A}^{[0]}$ and $\hat{A}^{[1]}$, with respect to x, vanish exactly at $x=1$, but we do not know this in advance.]

Therefore, with all these ordering arguments in mind, it becomes clear that the magnetic-field disturbance may be separated into an algebraic and an exponential contribution, $\hat{A}$ and $\hat{E}$, respectively. Moreover the contribution $\hat{E}^{[0]}$ appears at a higher order in the R expansion so the analysis of $\hat{A}^{[0]}$ can be performed independently. In fact, $\hat{A}^{[0]}$ is the most important part of the solution for large values of R. Using the definition of $\hat{A}$, and the expansion Eq. (73) in Eq. (62), we obtain that $\hat{A}^{[0]}$ satisfies the second-order differential equation

$$x\left[1-\frac{x}{2}\right]\frac{d^2\hat{A}^{[0]}}{dx^2} + (1-x)\frac{d\hat{A}^{[0]}}{dx} - \frac{k^2}{2}(x-\mu_1)(\mu_2 - x)\hat{A}^{[0]} = 0 \; . \tag{75}$$

Here we have introduced the roots of the coefficient multiplying $\hat{A}^{[0]}$, as

$$\mu_1 = 1 + \frac{1}{(\Gamma^{[0]})^2} - \left[1 + \frac{1}{(\Gamma^{[0]})^4}\right]^{1/2} \; ,$$
$$\mu_2 = 1 + \frac{1}{(\Gamma^{[0]})^2} + \left[1 + \frac{1}{(\Gamma^{[0]})^4}\right]^{1/2} \; . \tag{76}$$

Moreover, the boundary conditions of Eqs. (63) and (64), to leading order in the R expansion, require that

$$\hat{A}^{[0]} = 0 \quad \text{at } x=1 \; , \tag{77}$$

and

$$\hat{A}^{[0]}(x=0) \text{ remains finite} \; . \tag{78}$$

Effectively $\hat{A}^{[0]}$ is the solution for infinite conductivity (i.e., for the "case" $R=\infty$) which obeys a second-order differential equation and needs to satisfy only two boundary conditions instead of four. The other two boundary conditions involving $d^2\hat{b}/dx^2$ in Eqs. (63) and (64) are not needed to lowest order in $1/R$, consistently with the reduction in the order of the equation. The solution $\hat{A}^{[0]}$ will be studied in detail in the following sections. The contribution $\hat{E}^{[0]}$ is determined in the Appendix. Imposing that $\hat{A}^{[0]}$ is a nontrivial solution of Eq. (75), with the boundary conditions of Eqs. (77) and (78), leads to the dispersion relation for the disturbances which provides the leading order of the growth rate, $\Gamma^{[0]}$, as a function of the disturbance wave number k.

VII. THE SOLUTION $\hat{A}^{[0]}$

Note that the roots μ_1 and μ_2 of Eq. (76) are approximately given by

$$\mu_1 \simeq \frac{1}{(\Gamma^{[0]})^2} - \frac{1}{2(\Gamma^{[0]})^4} \quad \text{and} \quad \mu_2 \simeq 2 + \frac{1}{(\Gamma^{[0]})^2} \quad \text{when } \Gamma^{[0]} \gg 1 , \tag{79}$$

while

$$\mu_1 \simeq 1 - \frac{(\Gamma^{[0]})^2}{2} \quad \text{and} \quad \mu_2 \simeq \frac{2}{(\Gamma^{[0]})^2} + 1 \quad \text{when } \Gamma^{[0]} \ll 1 . \tag{80}$$

In general, for any value of $\Gamma^{[0]}$, we have $0 \le \mu_1 \le 1$ and $\mu_2 > 1$. Therefore, in the interval of interest, $x \in [0,1]$, the coefficient multiplying $\hat{A}^{[0]}$ in Eq. (75) vanishes only at one point, $x = \mu_1$. Moreover, the coefficient multiplying the highest derivative vanishes at the origin. Thus Eq. (75) has the following properties: the point $x = 0$ is a regular singular point and the point $x = \mu_1$ is a turning point. Away from these points the equation is regular, admitting regular expansions in the only parameter remaining, the wave number k. To analyze Eq. (75) we divide the interval [0,1] into four regions: (i) the region $1 \ge x > \mu_1$, (ii) the region $x \approx \mu_1$, in the vicinity of the turning point, (iii) the region $\mu_1 > x > 0$, and (iv) the region $x \approx 0$, the neighborhood of the singular point.

In the following we will provide an asymptotic analysis of Eq. (75) for $k \gg 1$ by studying the function $\hat{A}^{[0]}$ in each of the above four regions and matching the solution in each region with the solution in the adjacent regions. For a complete description of the matching of asymptotic expansions, and its use in equations with turning points and singular points, we refer to the book of Nayfeh [11].

A. Region I: $\mu_1 < x < 1$

The solution in this region I, when $k \gg 1$, may be expanded in terms of k. Following the WKB method, a particular solution of Eq. (75) may be obtained in the form of the asymptotic expansion

$$\hat{A}_{\mathrm{I}}^{[0]} = \left[1 + \sum_{n=1}^{\infty} f_n(x) k^{-n}\right] \exp\left[k h(x) + \sum_{n=0}^{\infty} g_n(x) k^{-n}\right] . \tag{81}$$

Substituting this expansion in Eq. (75) we get, to leading order $[O(k^2)]$,

$$x\left[1 - \frac{x}{2}\right]\left[\frac{dh}{dx}\right]^2 - \frac{1}{2}(\mu_2 - x)(x - \mu_1) = 0 ,$$

which has two independent solutions, $h_1 = h$ and $h_2 = -h$, with

$$h(x) = \int_{\mu_1}^{x} \left[\frac{(\mu_2 - t)(t - \mu_1)}{2t(1 - t/2)}\right]^{1/2} dt . \tag{82}$$

To order k we get

$$x\left[1 - \frac{x}{2}\right]\left[\frac{d^2 h_i}{dx^2} + 2\frac{dh_i}{dx}\frac{dg_0}{dx}\right] + (1 - x)\frac{dh_i}{dx} = 0 ,$$

which provides the solution for g_0,

$$g_0 = -\ln\left[\frac{(\mu_2 - x)(x - \mu_1) x (1 - x/2)}{2}\right]^{1/4} , \tag{83}$$

which is the same for the two solutions h_1 and h_2. The general solution in region I of Eq. (75) is a linear combination of the two independent solutions, and to leading order in k may be written in the form

$$\hat{A}_{\mathrm{I}}^{[0]} = \frac{c_1 2^{1/4} \exp[k h(x)]}{[(\mu_2 - x)(x - \mu_1) x (1 - x/2)]^{1/4}} + \frac{c_2 2^{1/4} \exp[-k h(x)]}{[(\mu_2 - x)(x - \mu_1) x (1 - x/2)]^{1/4}} , \tag{84}$$

with $h(z)$ defined by Eq. (82). The two independent contributions $\hat{b}_1^{[0]}$ and $\hat{b}_2^{[0]}$ of Eq. (73) are evident in Eq. (84).

The magnetic-field disturbance must vanish at the plasma-solid surface, as indicated by the boundary condition of Eq. (77). This condition relates the value of the two coefficients, c_1 and c_2,

$$c_2 = -c_1 e^{2kh(1)} . \tag{85}$$

Near the turning point the function $h(x)$ may be approximated by

$$h(x \to \mu_1) \simeq \int_{\mu_1}^{x} \left[\frac{\mu_2 - \mu_1}{\mu_1(2 - \mu_1)}\right]^{1/2} (t - \mu_1)^{1/2} dt = \frac{2}{3}\left[\frac{\mu_2 - \mu_1}{\mu_1(2 - \mu_1)}\right]^{1/2} (x - \mu_1)^{3/2} .$$

Therefore the solution in region I in the vicinity of the turning point $(x \to \mu_1)$ is approximately

$$\hat{A}_{\mathrm{I}}^{[0]} \simeq \frac{c_1 2^{1/2} (x - \mu_1)^{-1/4}}{[(\mu_2 - \mu_1)\mu_1(2 - \mu_1)]^{1/4}} \left\{\exp\left[k\frac{2}{3}(x - \mu_1)^{3/2}\left[\frac{\mu_2 - \mu_1}{\mu_1(2 - \mu_1)}\right]^{1/2}\right] - \exp\left[2kh(1) - k\frac{2}{3}(x - \mu_1)^{3/2}\left[\frac{\mu_2 - \mu_1}{\mu_1(2 - \mu_1)}\right]^{1/2}\right]\right\} , \tag{86}$$

valid for $x = \mu_1 \ll 1$. This limiting behavior should match with the solution in region II, near the turning point.

B. Region II, $x \approx \mu_1$

Near the turning point, $x = \mu_1$, and for large values of the wave number k, we define the inner variable

$$\xi = \left[\frac{\mu_2 - \mu_1}{\mu_1(2-\mu_1)} \right]^{1/3} k^{2/3}(x-\mu_1) , \tag{87}$$

in terms of which, and to leading order in k, Eq. (75) reduces to

$$\frac{d^2 \hat{A}_{\rm II}^{[0]}}{d\xi^2} - \xi \hat{A}_{0,\rm II} = 0 ,$$

whose general solution is [12]

$$\hat{A}_{\rm II}^{[0]} = c_{1,\rm II}\,{\rm Ai}(\xi) + c_{2,\rm II}\,{\rm Bi}(\xi) , \tag{88}$$

where $\mathrm{Ai}(\xi)$ and $\mathrm{Bi}(\xi)$ are the Airy functions of the first and second type, respectively, and $c_{1,\rm II}$ and $c_{2,\rm II}$ are two constants. For large *positive* values ξ, the asymptotic limits of the Airy functions are

$$\begin{aligned}
{\rm Ai}(\xi) &\approx \frac{1}{2\pi^{1/2}} \xi^{-1/4} \exp\left[-\frac{2}{3}\xi^{3/2} \right] , \\
{\rm Ai}(-\xi) &\approx \frac{1}{\pi^{1/2}} \xi^{-1/4} \sin\left[\frac{2}{3}\xi^{3/2} + \frac{\pi}{4} \right] , \\
{\rm Bi}(\xi) &\approx \frac{1}{\pi^{1/2}} \xi^{-1/4} \exp\left[\frac{2}{3}\xi^{3/2} \right] , \\
{\rm Bi}(-\xi) &\approx \frac{1}{\pi^{1/2}} \xi^{-1/4} \cos\left[\frac{2}{3}\xi^{3/2} + \frac{\pi}{4} \right] .
\end{aligned} \tag{89}$$

Thus, for large and positive values of the inner variable ξ, the solution in region II, given by Eq. (88), can be written in terms of the original variable x as

$$\begin{aligned}
\hat{A}_{\rm II}^{[0]} \approx{} & \frac{c_{1,\rm II}}{2\pi^{1/2}} \left[\frac{\mu_1(2-\mu_1)}{\mu_2-\mu_1} \right]^{1/12} \frac{k^{-1/6}}{(x-\mu_1)^{1/4}} \exp\left\{ -k\frac{2}{3}(x-\mu_1)^{3/2} \left[\frac{\mu_2-\mu_1}{\mu_1(2-\mu_1)} \right]^{1/2} \right\} \\
& + \frac{c_{2,\rm II}}{\pi^{1/2}} \left[\frac{\mu_1(2-\mu_1)}{\mu_2-\mu_1} \right]^{1/12} \frac{k^{-1/6}}{(x-\mu_1)^{1/4}} \exp\left\{ k\frac{2}{3}(x-\mu_1)^{3/2} \left[\frac{\mu_2-\mu_1}{\mu_1(2-\mu_1)} \right]^{1/2} \right\} ,
\end{aligned} \tag{90}$$

valid for

$$\xi = \left[\frac{\mu_2-\mu_1}{\mu_1(2-\mu_1)} \right]^{1/3} k^{2/3}(x-\mu_1) \gg 1 ,$$

but $(x-\mu_1) \ll 1$.

Matching the solutions in regions I and II of Eqs. (86) and (90) fixes the coefficients $c_{1,\rm II}$ and $c_{2,\rm II}$ in terms of c_1,

$$\begin{aligned}
c_{2,\rm II} &= \frac{(2\pi)^{1/2}k^{1/6}}{[\mu_1(2-\mu_1)]^{1/3}[\mu_2-\mu_1]^{1/6}} c_1 , \\
c_{1,\rm II} &= -2c_{2,\rm II}e^{2kh(1)} = -\frac{2(2\pi)^{1/2}k^{1/6}e^{2kh(1)}}{[\mu_1(2-\mu_1)]^{1/3}[\mu_2-\mu_1]^{1/6}} c_1 .
\end{aligned} \tag{91}$$

Using Eq. (89) we see that the solution Eq. (88) on the other side of the turning point $(x<\mu_1)$ behaves as

$$\begin{aligned}
\hat{A}_{\rm II}^{[0]} \approx{} & \frac{c_1 2^{1/2}(\mu_1-x)^{-1/4}}{[\mu_1(2-\mu_1)(\mu_2-\mu_1)]^{1/4}} \left\{ \cos\left[k\frac{2}{3} \left[\frac{\mu_2-\mu_1}{\mu_1(2-\mu_1)} \right]^{1/2} (\mu_1-x)^{3/2} + \frac{\pi}{4} \right] \right. \\
& \left. - 2e^{2kh(1)} \sin\left[k\frac{2}{3} \left[\frac{\mu_2-\mu_1}{\mu_1(2-\mu_1)} \right]^{1/2} (\mu_1-x)^{3/2} + \frac{\pi}{4} \right] \right\} ,
\end{aligned} \tag{92}$$

valid for $-\xi \gg 1$, but $(\mu_1 - x) \ll 1$. In order to obtain the previous relation, we have referred the coefficients to c_1 using Eq. (91). The behavior indicated by Eq. (92) should match with the limiting behavior in region III near the turning point. But before analyzing the solution in region III we look for the simpler region IV near the singularity at the origin.

C. Region IV

In the vicinity of the regular singular point at $x=0$, we define the coordinate

$$\zeta = (2\mu_1\mu_2 x)^{1/2} k \ , \tag{93}$$

which for large wave numbers k is a stretched variable. Then, to leading order in k, Eq. (75) reduces to

$$\zeta^2 \frac{d^2 \hat{A}_{\rm IV}^{[0]}}{d\zeta^2} + \zeta \frac{d\hat{A}_{\rm IV}^{[0]}}{d\zeta} + \zeta^2 \hat{A}_{\rm IV}^{[0]} = 0 \ . \tag{94}$$

The general solution of Eq. (94) is a linear combination of the Bessel functions of the first and second kind, $J_0(\zeta)$ and $Y_0(\zeta)$, respectively [13]. But $Y_0(\zeta)$ presents a logarithmic divergence at the origin. Therefore it cannot appear in the solution because of the boundary condition at $x=0$ of Eq. (78). Therefore we have

$$\hat{A}_{\rm IV}^{[0]} = c_{\rm IV} J_0(\zeta) \ . \tag{95}$$

For large values of ζ, and in terms of the original variable x, $A_{\rm IV}^{[0]}$ behaves as

$$\hat{A}_{\rm IV}^{[0]} \approx c_{\rm IV} \left[\frac{2}{\pi} \right]^{1/2} \frac{k^{-1/2}}{(2\mu_1\mu_2 x)^{1/4}} \cos \left[k(2\mu_1\mu_2 x)^{1/2} - \frac{\pi}{4} \right] , \tag{96}$$

valid for $\zeta = (2\mu_1\mu_2 x)^{1/2} k \gg 1$, but $x \ll 1$.

D. Region III, $0 < x < \mu_1$

In this region, for large k, an expansion of the solution may be written as

$$\hat{A}_{\rm III}^{[0]} = c_{\rm III} \left[\sum_{n=0}^{\infty} f_n(x) k^{-n} \right] \times \cos \left[km(x) - \frac{\pi}{4} + \sum_{n=0}^{\infty} g_n(x) k^{-n} \right] . \tag{97}$$

A corresponding value with a *sine* instead of a *cosine* should also be added to obtain the general solution, but it is easy to see that in order to match with the solution of Eq. (96) in region IV, the coefficient of this *sine* solution must vanish. This is the reason for our particular choice of the phase shift $-\pi/4$ in Eq. (97).

Using this expansion in Eq. (75) to leading order (order k^2) in a k expansion, the equation reduces to

$$-\left[\frac{dm}{dx} \right]^2 x \left[1 - \frac{x}{2} \right] + \frac{1}{2}(\mu_1 - x)(\mu_2 - x) = 0 \ ,$$

whose solution is

$$m(x) = \int_0^x \left[\frac{(\mu_1 - t)(\mu_2 - t)}{2t(1 - t/2)} \right]^{1/2} dt \ . \tag{98}$$

On the other hand, to order k, Eq. (75) becomes

$$x \left[1 - \frac{x}{2} \right] \left[2\frac{dm}{dx}\frac{df_0}{dx} + f_0 \frac{d^2 m}{dx^2} \right] + (1 - x) f_0 \frac{dm}{dx} = 0 \ ,$$

which may easily be integrated

$$f_0 = \left[\frac{2}{x(1 - x/2)(\mu_1 - x)(\mu_2 - x)} \right]^{1/4} . \tag{99}$$

Therefore, to leading order, the expansion of Eq. (97) becomes

$$\hat{A}_{\rm II}^{[0]} = \frac{c_{\rm III} 2^{1/4}}{[x(1 - x/2)(\mu_1 - x)(\mu_2 - x)]^{1/4}} \times \cos \left[km(x) - \frac{\pi}{4} \right] , \tag{100}$$

with $m(x)$ given by Eq. (98). The limiting behavior of $m(x)$ when x approaches the turning point at $x=\mu_1$ from below is

$$m(x \to \mu_1) \approx m(\mu_1) - \int_x^{\mu_1} \left[\frac{\mu_2 - \mu_1}{2\mu_1(1 - \mu_1/2)} \right]^{1/2} \times (\mu_1 - t)^{1/2} dt = m(\mu_1) - \frac{2}{3} \left[\frac{\mu_2 - \mu_1}{\mu_1(2 - \mu_1)} \right]^{1/2} (\mu_1 - x)^{3/2} \ .$$

Therefore the limiting behavior of $\hat{A}_{\rm III}^{[0]}$ near the turning point is

$$\hat{A}_{\rm III}^{[0]}(x \to \mu_1) \approx \frac{c_{\rm III} 2^{1/2} (\mu_1 - x)^{-1/4}}{[\mu_1(2 - \mu_1)(\mu_2 - \mu_1)]^{1/4}} \left\{ \cos \left[k\frac{2}{3} \left[\frac{\mu_2 - \mu_1}{\mu_1(2 - \mu_1)} \right]^{1/2} (\mu_1 - x)^{3/2} + \frac{\pi}{4} \right] \cos[km(\mu_1)] + \sin \left[k\frac{2}{3} \left[\frac{\mu_2 - \mu_1}{\mu_1(2 - \mu_1)} \right]^{1/2} (\mu_1 - x)^{3/2} + \frac{\pi}{4} \right] \sin[km(\mu_1)] \right\} . \tag{101}$$

Comparing this expression with the behavior of $\hat{A}_{\rm II}^{[0]}$ far from the turning point (far, from the inner stretched variable point of view), given by Eq. (92), we get

$$c_{\rm III} = \frac{c_1}{\cos[km(\mu_1)]} \ , \tag{102}$$

and also

$$\tan[km(\mu_1)] = -2\exp[2kh(1)] \ . \tag{103}$$

This relation, with the functions m and h given by Eqs. (98) and (82), is the *dispersion relation* for the lowest-order growth rate $\Gamma^{[0]}$ and the wave number k. Equation (103) is one of the main results of the present paper. The analysis of this dispersion relation is deferred to the next section.

For very small values of x (near the singular point at $x=0$), the solution in region III, taking into account Eqs. (100) and (102), behaves as

$$\hat{A}^{[0]}_{\rm III} \approx \frac{c_1}{\cos[km(\mu_1)]}\left[\frac{2}{\mu_1\mu_2}\right]^{1/4} x^{-1/4}\cos\left[k(2\mu_1\mu_2 x)^{1/2}-\frac{\pi}{4}\right] , \tag{104}$$

which matches with the solution in the vicinity of $x=0$, Eq. (96), for

$$c_{\rm IV} = \frac{\pi^{1/2}k^{1/2}}{\cos[km(\mu_1)]}c_1 \ . \tag{105}$$

We have finished matching the first order of the asymptotic expansions in the four regions, for large values of k, relating all the coefficients to c_1, which remains as the factor controlling the initial amplitude of the disturbances.

E. Composite solutions and dispersion relation

As it is customary in asymptotic expansion analysis, we now proceed to write the composite solutions as combinations of the solutions in each region. We still have to distinguish between two regions connected at the turning point, $x=\mu_1$. For $x \ge \mu_1$ the leading-order solution of Eq. (75) may be written as the sum of the solutions in region I of Eq. (84), with c_2 given by Eq. (85), and the solution in region II of Eq. (88), with $c_{2,\rm II}$ and $c_{1,\rm II}$ given by Eq. (91). The common part of both solutions, either Eq. (86) or Eq. (90), must be subtracted to get

$$\begin{aligned}\hat{A}^{[0[} = {} & c_1 2^{1/4}\frac{\exp[kh(x)]-\exp\{k[2h(1)-h(x)]\}}{[(\mu_2-x)(x-\mu_1)x(1-x/2)]^{1/4}} \\ & + c_1\frac{(2\pi)^{1/2}k^{1/6}}{[\mu_1(2-\mu_1)]^{1/3}[\mu_2-\mu_1]^{1/6}}\left\{\mathrm{Bi}\left[\left[\frac{\mu_2-\mu_1}{\mu_1(2-\mu_1)}\right]^{1/3}k^{2/3}[x-\mu_1]\right]-2e^{2kh(1)}\mathrm{Ai}\left[\left[\frac{\mu_2-\mu_1}{\mu_1(2-\mu_1)}\right]^{1/3}k^{2/3}[x-\mu_1]\right]\right\} \\ & - c_1\frac{2^{1/2}(x-\mu_1)^{-1/4}}{[(\mu_2-\mu_1)\mu_1(2-\mu_1)]^{1/4}}\left\{\exp\left[k\frac{2}{3}(x-\mu_1)^{3/2}\left[\frac{\mu_2-\mu_1}{\mu_1(2-\mu_1)}\right]^{1/2}\right]\right. \\ & \left. - \exp\left[2kh(1)-k\frac{2}{3}(x-\mu_1)^{3/2}\left[\frac{\mu_2-\mu_1}{\mu_1(2-\mu_1)}\right]^{1/2}\right]\right\} ,\end{aligned} \tag{106}$$

valid for $x \ge \mu_1$, with $h(x)$ given by Eq. (82).

Another composite solution may be written for the whole region $x \le \mu_1$ by adding the solution in region III of Eq. (100), with $c_{\rm III}$ given by Eq. (102), to the solution in region IV of Eq. (95), with $c_{\rm IV}$ provided by Eq. (105). The common part of either Eq. (96) or Eq. (104) must be subtracted to get

$$\begin{aligned}\hat{A}^{[0]} = \frac{c_1}{\cos[km(\mu_1)]} & \left\{\left[\frac{2}{x(1-x/2)(\mu_1-x)(\mu_2-x)}\right]^{1/4}\cos\left[km(x)-\frac{\pi}{4}\right]+\pi^{1/2}k^{1/2}J_0[(2\mu_1\mu_2 x)^{1/2}k]\right. \\ & \left. -\left[\frac{2}{\mu_1\mu_2 x}\right]^{1/4}\cos\left[(2\mu_1\mu_2 x)^{1/2}k-\frac{\pi}{4}\right]\right\} ,\end{aligned} \tag{107}$$

valid for $x \le \mu_1$, with $m(x)$ given by Eq. (98).

Now the change undergone by the solution at the turning point becomes clear. Whereas for $x \le \mu_1$ Eq. (107) shows an oscillatory behavior, for $x \ge \mu_1$ its behavior is dominated by the exponential terms in Eq. (106). Notice that although we have only imposed two boundary conditions on $\hat{A}^{[0]}$, in fact it satisfies also the other two boundary conditions which are required for the magnetic-field disturbance $\hat{b}$, by Eqs. (63) and (64). Since $\hat{A}^{[0]}$ verifies all the boundary conditions, the remaining exponential contribution $\hat{E}$ plays only a minor role, as will be shown in the Appendix.

The results obtained so far for $\hat{A}^{[0]}$ are valid to leading order in an expansion for large values of the wave number k. But as we will see, in fact, they provide a good approximation for the whole range of k.

For small values of k and $\Gamma^{[0]}$, μ_1 and μ_2 are given by Eq. (80). It is easy to see that $h(1) \approx (\Gamma^{[0]})^2/3$. Then $kh(1)$ is very small and we may take, from the dispersion relation Eq. (103), the limiting behavior

$$\tan[km(\mu_1)] \approx -2 \quad \text{when } \Gamma^{[0[} \ll 1 \ . \tag{108}$$

Also in this limit

$$m(\mu_1) \approx \int_0^{\mu_1} \mu_2^{1/2} \left[\frac{\mu_1 - x}{2x(1-x/2)} \right]^{1/2} dx \ ,$$

which may be written as

$$m(\mu_1) \approx \frac{\delta}{\Gamma^{[0]}} \ , \quad \text{with } \delta = 2^{1/2} \int_0^1 \left[\frac{1-t}{t(2-t)} \right]^{1/2} dt \ .$$

But

$$\int_t^1 \left[\frac{1-t}{t(2-t)} \right]^{1/2} dt = 2^{3/2} E(\varphi|\tfrac{1}{2}) - 2^{1/2} F(\varphi|\tfrac{1}{2}) - 2\left[\frac{(1-t)t}{2-t} \right]^{1/2} ,$$

where E and F are the elliptic integrals [14] of the second and first kinds, respectively, and

$$\varphi = \arcsin \left[\frac{2(1-t)}{2-t} \right]^{1/2} .$$

Therefore, $\delta = 4E(\pi/2|\frac{1}{2}) - 2F(\pi/2|\frac{1}{2}) \approx 1.694$.

There exist different values of $\Gamma^{[0]}$ which satisfy Eq. (108). These values are such that $k\delta/\Gamma_n^{[0]} = \arctan(-2) \approx (n+0.6476)\pi$. That is,

$$\Gamma_n^{[0]} \approx \frac{1.694}{(n+0.6476)\pi} k \quad \text{for } n = 0,1,2,\ldots \quad \text{when } k \ll 1 \ . \tag{109}$$

Equation (109) shows that there exists an infinite number of unstable modes (one for each natural value of n). The value of $\Gamma_n^{[0]}$ is positive for any wave number k and any initially small disturbance of this kind will increase its amplitude exponentially with time as can be seen from Eq. (44). The mode that is most likely to be observed is the one with the fastest growing rate. This fastest growing mode $n=0$ corresponds to

$$\Gamma_0^{[0]} \approx 0.833k \quad \text{when } k \ll 1 \ . \tag{110}$$

On the other hand, for large values of k (corresponding to large values of $\Gamma^{[0]}$), μ_1 and μ_2 are approximately given by Eq. (79). From Eq. (82) it can be shown that $h(\Gamma^{[0]} \gg 1) \approx 1$, whereas from Eq. (98) there results

$$m(\mu_1) \approx \int_0^{\mu_1} \left[\frac{\mu_1 - x}{x} \right]^{1/2} \left[\frac{\mu_2}{2} \right]^{1/2} dx \quad \text{for } \Gamma^{[0]} \gg 1 \ .$$

This integral can be performed by a change of variable $x^{1/2} = \mu_1^{1/2} \sin t$. Then, using Eq. (79),

$$m(\mu_1) \approx \frac{\pi}{2(\Gamma^{[0]})^2} \quad \text{for } \Gamma^{[0]} \gg 1 \ .$$

Therefore, for large values of k and $\Gamma^{[0]}$, from the dispersion relation Eq. (103), we obtain

$$\tan[km(\mu_1)] \simeq \tan\left[\frac{\pi k}{2(\Gamma^{[0]})^2} \right] \to -\infty \ . \tag{111}$$

There are again different values of $\Gamma^{[0]}$ which satisfy this condition. They are such that

$$(\Gamma_n^{[0]})^2 \approx \frac{k}{2n+1} \quad \text{for } n = 0,1,2,\ldots \text{ when } k \gg 1 \ .$$

This equation provides the dispersion relation for large-wave number disturbances. One can improve this result by obtaining the next correction in the $1/k$ expansion. After some lengthy but straightforward analysis, using the next correction to $m(\mu_1)$ in Eq. (111), one obtains that

$$(\Gamma_n^{[0]})^2 \approx \frac{k}{2n+1} - \frac{1}{4} \quad \text{for } k \gg 1 \ . \tag{112}$$

The fastest growing mode, $n=0$, corresponds to

$$(\Gamma_0^{[0]})^2 \approx k - \tfrac{1}{4} \quad \text{for } k \gg 1 \ , \tag{113}$$

which, to leading order, coincides with the well-known dispersion relation of the Rayleigh-Taylor instability of Eq. (1). Since $\Gamma_0^{[0]}$ is a continuously increasing function of the wave number k, and it is positive, the disturbances with large wave number (shorter wavelengths) will grow faster. The above results are valid to lowest order in R. In fact, the shorter wavelengths (which involve large gradients of the magnetic-field disturbances) should be stabilized by the magnetic diffusivity.

Equation (109) (for $k \ll 1$) and Eq. (112) (for $k \gg 1$) are much simpler than the complete dispersion relation Eq. (103) and allow an easy approximate determination of the growth rate $\Gamma^{[0]}$ in each case. To check the validity of Eqs. (109) and (112) we compare them with the numerical results obtained by Powell [5]. Powell provided some numerical values of the growth rate in the limit $R=\infty$ but Λ remaining finite. Denoting by subscript P the variables in Powell's paper, we get $\Gamma^{[0]} = \bar{\omega}_P \Lambda^{1/2}$ and $\Lambda = 1/\chi_P$. Powell presented results for $\beta_P = 1$, which corresponds to $\chi_P = 1.092$ and $\Lambda = 0.916$, for $\beta_P = 0.5$ ($\chi_P = 0.6249$, $\Lambda = 1.6$), and for $\beta_P = 0.1$ (χ_P

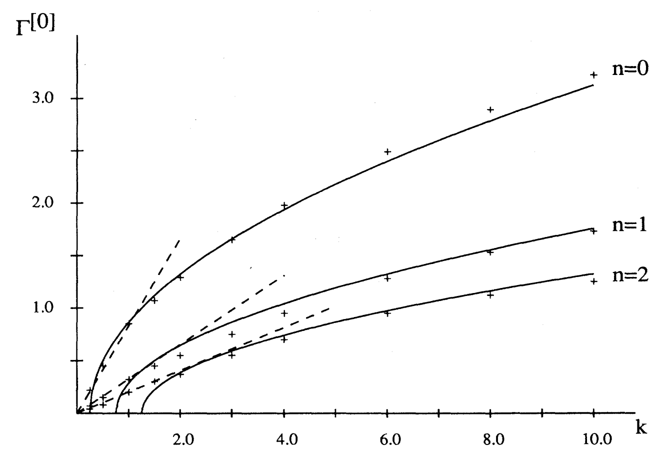

FIG. 2. Dimensionless growth rate $\Gamma^{[0]}$ of the three most unstable modes ($n=0$, 1, and 2) versus the dimensionless wave number k. The solid and broken lines are our asymptotic ($\Lambda \to \infty$, $R \to \infty$) results. The solid lines correspond to Eq. (112) for $k \gg 1$ and the dashed lines to Eq. (109) for $k \ll 1$. The crosses come from the numerical results presented in Fig. 3 of Ref. [5] for $\Lambda = 0.916$.

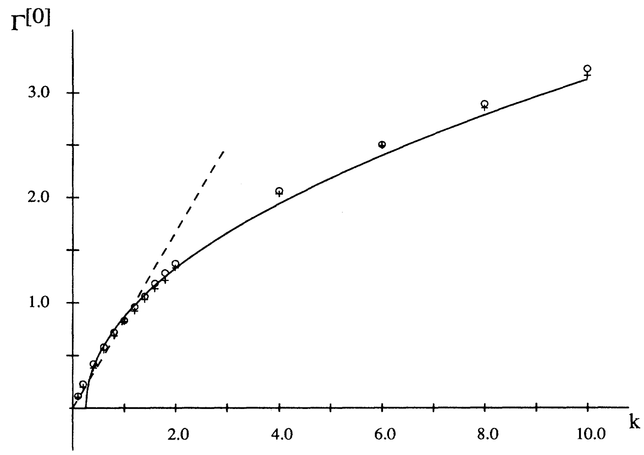

FIG. 3. Dimensionless growth rate $\Gamma^{[0]}$ versus dimensionless wave number k for the fastest growing mode ($n=0$). Solid and dashed lines are our asymptotic ($\Lambda\to\infty$, $R\to\infty$) results, Eq. (113) for $k\gg 1$ and Eq. (110) for $k\ll 1$, respectively. The circles and crosses come from the numerical results presented in Fig. 4 of Ref. [5], for $\Lambda=6.95$ and 1.6, respectively.

$=0.1438$, $\Lambda=6.95$). Figure 2 represents the growth rate $\Gamma^{[0]}$ versus the disturbance wave number k of the three fastest growing modes for $\Lambda=0.916$. The solid lines correspond to Eq. (112) and the broken lines to Eq. (109), whereas the crosses were obtained in Fig. 3 in Powell's paper. Even though the value of Λ is far from being large, and our analysis was restricted to $\Lambda=\infty$, the agreement between Powell's numerical results and our analytical results is quite good. To further emphasize this agreement we plot in Fig. 3 our results for the fastest growing modes ($n=0$). The solid lines come from Eq. (113) and the broken lines from Eq. (110). The crosses and circles represent the numerical results presented by Powell in his Fig. 4. The crosses are for $\Lambda=6.95$ and the circles for $\Lambda=1.6$. There is little appreciable difference in the figures between our analytical results and the numerical results of Powell. The two curves of Powell's Fig. 4 fall almost on top of each other when plotted in our variables in Fig. 3, showing that we are using the proper variables to describe the problem.

VIII. THE CORRECTION DUE TO NONZERO RESISTIVITY

In this section we will show that the correction $\Gamma^{[1]}$ in the expansion for Γ of Eq. (66) is zero, and also obtain the first nonvanishing correction, $\Gamma^{[2]}$.

Using the expansion of Eq. (73) in Eq. (62), the equation for $\hat{A}^{[1]}$ reduces to

$$x\left[1-\frac{x}{2}\right]\frac{d^2\hat{A}^{[1]}}{dx^2}+(1-x)\frac{d\hat{A}^{[1]}}{dx}-\frac{k^2}{2}(x-\mu_1)\times(\mu_2-x)\hat{A}^{[1]}=\frac{\Gamma^{[1]}}{(\Gamma^{[0]})^3}2k^2(1-x)\hat{A}^{[0]}\,, \qquad (114)$$

whereas the boundary conditions, to order $O(R^{-1/2})$, become

$$\hat{A}^{[1]}(x=1)=0 \quad\text{and}\quad \hat{A}^{[1]}(x=0)\ \text{remains finite}\,. \qquad (115)$$

This is an inhomogeneous equation with the same operator on the left side as in Eq. (75), and exactly the same boundary conditions which applied to $\hat{A}^{[0]}$. As usual, by the adjoint theorem, the existence of a nontrivial solution for the inhomogeneous system requires as a compatibility condition that the inhomogeneous part be orthonormal to the solution of the adjoint homogeneous problem. It can be shown that the operator in the equation for $\hat{A}^{[0]}$ is self-adjoint. Therefore the right-hand side of Eq. (114) should be orthonormal to $\hat{A}^{[0]}$.

$$\frac{2k^2\Gamma^{[1]}}{(\Gamma^{[0]})^3}\int_0^1(1-x)(\hat{A}^{[0]})^2dx=0\,.$$

This condition is fulfilled only when

$$\Gamma^{[1]}=0\,. \qquad (116)$$

As a consequence, the expansion for Γ in terms of $1/R$ becomes

$$\Gamma=\Gamma^{[0]}+\sum_{n=2}^{\infty}\Gamma^{[n]}R^{-n/2}\,, \qquad (117)$$

with the dependence of $\Gamma^{[0]}$ on the wave number obtained from Eq. (103). The equation for $\hat{A}^{[1]}$ becomes the same as for $\hat{A}^{[0]}$, and therefore both are proportional; that is,

$$\hat{A}^{[1]}=\text{const}\times\hat{A}^{[0]}\,, \qquad (118)$$

where the value of the proportionality constant would be determined by the solution of the equation for $\hat{E}^{[1]}$.

The inhomogeneous differential equation for the next order correction $\hat{A}^{[2]}$ is

$$x\left[1-\frac{x}{2}\right]\frac{d^2\hat{A}^{[2]}}{dx^2}+(1-x)\frac{d\hat{A}^{[2]}}{dx}-\frac{k^2}{2}(x-\mu_1)(\mu_2-x)\hat{A}^{[2]}$$
$$=\frac{\Gamma^{[2]}}{(\Gamma^{[0]})^3}2k^2(1-x)\hat{A}^{[0]}+\frac{1}{\Gamma^{[0]}}\left\{\frac{d^4\hat{A}^{[0]}}{dx^4}F+2\frac{dF}{dx}\frac{d^3\hat{A}^{[0]}}{dx^3}+\frac{d^2\hat{A}^{[0]}}{dx^2}\left[\frac{d^2F}{dx^2}-2Fk^2\right]\right.$$
$$\left.-2k^2\frac{dF}{dx}\frac{d\hat{A}^{[0]}}{dx}+\hat{A}^{[0]}\left[k^4F-k^2\frac{d^2F}{dx^2}\right]\right\}\,. \qquad (119)$$

The boundary conditions of Eqs. (63) and (64), taking into account the result of the Appendix which shows that the leading-order contribution of the exponential term $\hat{E}$ is of order R^{-2}, reduce again to

$$\hat{A}^{[2]}(x=1)=0 \quad \text{and} \quad \hat{A}^{[2]}(x=0) \text{ remains finite} . \tag{120}$$

The compatibility condition for the existence of a nontrivial solution of Eq. (119) with boundary conditions Eq. (120) is obtained by multiplying Eq. (119) by $\hat{A}^{[0]}$, and integrating over the whole interval in x from 0 to 1. This compatibility condition provides the value of $\Gamma^{[2]}$. Since the operator of the homogeneous equation is self-adjoint we get

$$\Gamma^{[2]}=-\frac{(\Gamma^{[0]})^2}{2k^2}\frac{\Phi_1(k)}{\Phi_2(k)} . \tag{121}$$

The functions $\Phi_i(k)$ are given by

$$\Phi_1(k)=\int_0^1 \hat{A}^{[0]}\left\{\frac{d^4\hat{A}^{[0]}}{dx^4}F+2\frac{dF}{dx}\frac{d^3\hat{A}^{[0]}}{dx^3} +\frac{d^2\hat{A}^{[0]}}{dx^2}\left[\frac{d^2F}{dx^2}-2Fk^2\right] -2k^2\frac{dF}{dx}\frac{d\hat{A}^{[0]}}{dx} +\hat{A}^{[0]}\left[k^4F-k^2\frac{d^2F}{dx^2}\right]\right\}dx ,$$

and

$$\Phi_2(k)=\int_0^1(1-x)(\hat{A}^{[0]})^2dx . \tag{122}$$

We can write $\Phi_1(k)$ in the form

$$\Phi_1=\int_0^1\hat{A}^{[0]}\left[\frac{d^2}{dx^2}\left[F\frac{d^2\hat{A}^{[0]}}{dx^2}\right]-k^2\frac{d^2}{dx^2}(F\hat{A}^{[0]}) -k^2F\frac{d^2\hat{A}^{[0]}}{dx^2}+k^4F\hat{A}^{[0]}\right]dx .$$

Integrating twice by parts the first and second terms in the right-hand side, and taking into account Eqs. (60) and (75), we get

$$\Phi_1=\int_0^1\frac{(1-x)^2}{(1-x)^2+x(1-x/2)} \times\left[\frac{d\hat{A}^{[0]}}{dx}+\frac{k^2}{(\Gamma^{[0]})^2}\hat{A}^{[0]}\right]^2dx . \tag{123}$$

Written in these terms, it becomes clear that Φ_1 and Φ_2 are both *positive definite*. Thus the magnetic diffusivity introduces a term $\Gamma^{[2]}$ which, from Eq. (121), is *always negative* and tends to stabilize the plasma. Thus the diffusive effects have a stabilizing role on the plasma, as may be expected for any dissipative phenomenon.

The limiting behavior of Φ_1 and Φ_2 for large wave numbers k may be evaluated by the usual techniques of asymptotic expansion of integrals; see, for instance, Bender and Orszag [15]. For large values of k the function $\hat{A}^{[0]}$ has $n+1$ extrema, with n representing the mode number. These extrema are located between the turning point and the plasma-vacuum interface. For large values of k the turning point is located very close to the plasma-vacuum interface, and the extrema of $\hat{A}^{[0]}$ become very pronounced. Therefore the asymptotic limit of Φ_1 and Φ_2 will be given by the contribution to the integrals in the vicinity of these extrema. Using this one can show that

$$\Phi_1(k)\simeq\frac{k^4}{(\Gamma^{[0]})^4}\Phi_2(k) \quad \text{for } k\gg 1 .$$

Using this result in Eq. (121) we obtain

$$\Gamma_n^{[2]}\simeq-\frac{k^2}{2(\Gamma^{[0]})^2}\simeq-\frac{2n+1}{2}k \quad \text{for } k\gg 1 . \tag{124}$$

Here we have used the leading behavior of $\Gamma^{[0]}$ of Eq. (112) for $k\gg 1$. Equation (124) gives the first contribution (for large wave numbers k) to the growth rate due to the finiteness of the plasma resistivity. Therefore, from the expansion of Eq. (117), and for each mode n, we get

$$\Gamma_n\simeq\frac{k^{1/2}}{(2n+1)^{1/2}}-\frac{2n+1}{2}\frac{k}{R} \quad \text{for } R\gg 1 \text{ and } k\gg 1 . \tag{125}$$

This simple expression provides the asymptotic limiting behavior of the dispersion relation for any mode n. Equation (125) comprises most of the asymptotic results of the present analysis. The above result is rewritten in terms of the dimensional variables K and Ω in Eqs. (46) and (47) as

$$\Omega_n\simeq\left[\frac{aK}{2n+1}\right]^{1/2}-\frac{2n+1}{2}\frac{K}{\sigma\mu l} \quad \text{for } \sigma\mu(al^3)^{1/2}\gg 1 \text{ and } Kl\gg 1 , \tag{126}$$

where we used Eq. (3) for R.

In the case $n=0$ (the fastest growing mode), Eq. (125) reduces to

$$\Gamma_0\simeq k^{1/2}-\frac{k}{2R} \quad \text{for } R\gg 1 \text{ and } k\gg 1 . \tag{127}$$

This equation modifies the Rayleigh-Taylor dispersion relation of Eq. (1) by taking into account the first contribution of the diffusive effects, incorporated in the definition of the modified Reynolds number R. The stabilizing role of the plasma resistivity is more important for the larger-wave-number disturbances because these have large gradients in the magnetic field which are opposed by the magnetic diffusivity. Thus from Eq. (125) we see that Γ increases with the wave number k until it reaches a maximum value and thereafter decreases with k. The maximum growth rate, for each mode n, corresponds to

$$k_{n,\max}\simeq\frac{R^2}{(2n+1)^3} \quad \text{and} \quad \Gamma_{n,\max}\simeq\frac{R}{2(2n+1)^2} . \tag{128}$$

In a given experiment with an accelerated plasma one may expect that the disturbance with the highest growth rate will overcome all the other disturbances and become dominant in the long range. This disturbance will be characterized by the mode $n=0$ at its highest growth rate in Eq. (128), that is,

$$k_{0,\max} \simeq R^2 \quad \text{and} \quad \Gamma_{0,\max} \simeq \frac{R}{2} \ . \tag{129}$$

As we will see in the next section this result lies at the limit of the range of validity of our asymptotic analysis and therefore it is expected to provide only an order of magnitude estimate of the fastest growing disturbance. Thus, in order to completely delimit the region of validity of our results, the value of the wave number beyond which the magnetic diffusive effects become dominant will be indicated in the next section.

IX. RANGE OF VALIDITY OF THE RESULTS

The question remains open. How large should R be for these results to hold? We write Eq. (A13) of the Appendix as $\tilde{\eta}=(x/\delta_R)^{1/2}$, where $\delta_R=(4\Gamma^{[0]}R)^{-1}$ is the characteristic length associated with the boundary layer introduced by R finite. On the other hand, the singularity at $x=0$ in Eq. (75) for $R=\infty$ affects the boundary layer explored by Eq. (93). This may be written as $\zeta=(x/\delta_0)^{1/2}$ with $\delta_0=(\Gamma^{[0]}/2k)^2$. The singularity has influence over lengths of the order δ_0. [Note that from Eq. (76), $\mu_1\mu_2=2/(\Gamma^{[0]})^2$.] The results indicated in Sec. VII are valid as long as the boundary region introduced by R finite is well embedded inside the region affected by the singularity ($\delta_R \ll \delta_0$); that means for $k \ll [(\Gamma^{[0]})^3 R]^{1/2}$. But for large values of k we have found that the fastest growing mode corresponds to $\Gamma_0^{[0]}=k^{1/2}$. Then, the results are valid for $k \ll R^2$, and therefore $\Gamma^{[0]} \ll R$. The same validity criterion may be obtained by imposing that for large wave numbers k and large values of R, the contribution $\Gamma^{[2]}/R$ should be much smaller than the leading-order contribution $\Gamma^{[0]}$ in the expansion for Γ. From Eq. (125), this validity criterion leads to the conclusion that these asymptotic results are valid for

$$1 \ll k \ll R^2 \ . \tag{130}$$

On the other hand, in the region where

$$\frac{k^2}{R} \gg 1 \tag{131}$$

diffusion plays a dominant role. Here it is expected that the large-wave-number disturbances, which involve very large gradients, will be damped by diffusion. A preliminary analysis of this large k limit shows that the asymptotic behavior of the growth rate should be

$$\Gamma \simeq -\frac{k^2}{R\alpha} \ , \quad \text{with } \alpha > 1 \text{ when } k^2 \gg 1 \text{ and } k^2 \gg R \ . \tag{132}$$

The full analysis of this problem is almost as complicated as the work presented here and is left for a future paper.

When the above inequalities (130) and (131) do not hold we are in an intermediate region and the dispersion relation is given by neither Eq. (103) nor Eq. (132). A different analysis should be done in this intermediate range. Unfortunately it seems that a complete analytical treatment of this case is not feasible and we must rely on numerical methods to obtain the growth rate of the larger-wave-number disturbances. We now sketch the reason for this failure. When the previous analysis breaks down, one may rescale the variables by defining

$$k=\kappa R^2$$

and the new expansion

$$\Gamma=\theta R+\sum_{n=0}^{\infty}\Theta^{[n]}R^{2-n} \ . \tag{133}$$

With κ and θ of order unity it can be shown that to leading order in an R expansion the solution of Eq. (62) may be written as

$$\hat{b}=c_1\hat{b}_1(x)e^{\kappa R^2 x}+c_2\hat{b}_2(x)e^{\kappa R^2 x} +c_3\hat{b}_3(x)e^{-\kappa R^2 x}+c_4\hat{b}_4(x)e^{-\kappa R^2 x} \ , \tag{134}$$

where the functions $\hat{b}_i$ ($i=1,\ldots,4$), to leading order in R, should satisfy the differential equation

$$\frac{d}{dx}\left[F\frac{d\hat{b}_i}{dx}\right]\mp\frac{\theta}{2\kappa}\frac{d}{dx}\left[x\left[1-\frac{x}{2}\right]\hat{b}_i\right] +\frac{\theta}{4\kappa}\left[\pm 1-\frac{\kappa}{\theta^2}\right](1-x)\hat{b}_i=0 \ , \tag{135}$$

where the upper sign applies to $\hat{b}_1$ and $\hat{b}_2$ while the lower sign applies to $\hat{b}_3$ and $\hat{b}_4$. Moreover, $\hat{b}$ must verify the boundary conditions of Eq. (63) at $x=1$.

Again, the WKB expansion overlying the solution of Eq. (134) is not valid when the term appearing in the exponential becomes of order unity. Thus, near the origin, we define

$$\phi=kx=\kappa R^2 x \ .$$

In terms of this new stretched coordinate Eq. (62), to leading order in the R expansion, becomes

$$\frac{d^2}{d\phi^2}\left[\phi^2\frac{d^2\hat{b}}{d\phi^2}\right]-\frac{d}{d\phi}\left[\left[2\phi^2+\phi\frac{\theta}{\kappa}\right]\frac{d\hat{b}}{d\phi}\right] +\hat{b}\left[\phi^2+\phi\frac{\theta}{\kappa}-\frac{1}{\theta}-2\right]=0 \ . \tag{136}$$

The solution of this equation should verify the boundary conditions of Eq. (64), and for very large values of ϕ must match with the solution indicated in Eq. (134) when x goes to zero. The matching would provide the dispersion relation for the range of wave numbers and amplification factors indicated in Eq. (133). Unfortunately, it seems that an analytical solution of Eq. (135) or Eq. (136) is not feasible, and the only way of obtaining the normalized growth rate θ of Eq. (133) is by means of appropriate numerical methods.

X. DISCUSSION OF RESULTS

We have studied the Rayleigh-Taylor instability in an accelerating plasma arc driven by $\mathbf{J}\times\mathbf{B}$ forces as found in electromagnetic rail launchers. Assuming a two-dimensional geometry there exist only two dimensionless parameters which control the behavior of the plasma. One of the parameters, denoted by Λ in Eq. (2), is proportional to the ratio between the square of the plasma sound speed and the plasma acceleration. We have obtained analytical results in the limit of large Λ. This is appropriate for the relatively low accelerations found when the projectile mass is much greater than the plasma mass, as usual in rail launchers. The other remaining parameter is a modified Reynolds number R given by Eq. (3), which accounts for the relative importance of convective over diffusive transport of the plasma magnetic field. In the plasma arcs produced in electromagnetic rail launchers R usually seems to take reasonably large values. In consequence we have performed a WKB expansion of the equation governing the magnetic-field disturbances, Eq. (62), using $1/R$ as a smallness parameter. With this technique the solution of the fourth-order differential equation of Eq. (62) is separated into an algebraic contribution $\hat{A}$, given by Eq. (73), and an exponentially small contribution $\hat{E}$, given by Eq. (72). In Sec. VI it was shown that in the expansion for $\hat{A}$ the leading-order term $\hat{A}^{[0]}$ is the solution of the problem for a plasma with an infinite electrical conductivity (i.e., the solution for the problem with $R=\infty$), where the governing equation reduces to a second-order differential equation. The analysis of the leading-order terms of the exponential contribution $\hat{E}^{[0]}$ has been undertaken in the Appendix, showing that $\hat{E}^{[0]}$ is exponentially small.

The equation governing $\hat{A}^{[0]}$, Eq. (75), is complicated. It has a turning point at an intermediate location $x=\mu_1$, and a singular point at the plasma-vacuum interface at $x=0$. Equation (75) was analyzed in Sec. VII in the limit of large disturbance wave numbers k by dividing the interval of interest into four regions and performing a matching asymptotic analysis, with $1/k$ as the smallness parameter. The matching between the different regions gave the dispersion relation of Eq. (103) for $\Gamma^{[0]}(k)$ as $R\rightarrow\infty$. For very small wave numbers the dispersion relation is reduced to Eq. (109), showing that the amplification factor increases linearly with k. However, for large k the dispersion relation is approximately given by Eq. (112), and the growth rate increases as $k^{1/2}$. In both limits it is shown that there exists an infinite number of modes all of which render the plasma steady state unstable. For the fastest growing mode and large wave number the classical dispersion relation for the Rayleigh-Taylor instability is recovered, Eq. (113).

Even though the dispersion relation of Eq. (103) has been obtained in the limit of very large wave numbers, $k\gg 1$, and for an infinite value of the parameter Λ, it seems to provide an excellent approximation for a large range of wave numbers, even in the case of Λ of order unity. This has been demonstrated by comparing our results with the results previously obtained by Powell [5], who did a direct numerical integration of the equation for infinite electrical conductivity.

In Sec. VIII we have shown that for large but finite values of the Reynolds number R, the first nonvanishing correction for the amplification factor Γ is of order $1/R$. We have been able to express this contribution, $\Gamma^{[2]}$, in closed form in Eq. (121) (in terms of the solution $\hat{A}^{[0]}$), by imposing the compatibility condition of the operator affecting $\hat{A}^{[2]}$. We have shown that $\Gamma^{[2]}$ is always negative, so the electrical resistivity tends to stabilize the plasma, as one expects for a dissipative phenomenon. Equation (124) provides a simple expression for $\Gamma^{[2]}$, asymptotically valid in the limit of large k. The leading terms in the dispersion relation for large k and R are given in Eq. (125). This Eq. (125) includes the main asymptotic results of the present analysis. The disturbance with the highest growth rate is characterized by Eq. (129), and although this result is at the limit of validity of our asymptotic analysis it is expected to provide a quite good estimation of the fastest growing disturbance.

The asymptotic results obtained in this paper remain valid as long as the wave number k is not too large. In Sec. IX we established that the dispersion relation remains valid until k becomes of order R^2. For these very large wave numbers the magnetic diffusivity effects become dominant. They will tend to stabilize the plasma by smoothing out the short-wavelength magnetic-field inhomogeneities. This limit of large magnetic diffusivity should be the subject of further analysis in future work.

ACKNOWLEDGMENTS

This work was carried out while one of the authors (J. L. C.) was spending a sabbatical year at the Department of Physics, University of Miami. The financial support by Dirección General de Investigación Científica y Técnica (DGICYT), Ministerio de Educación y Ciencia (Spain), which made possible this sabbatical, is gratefully acknowledged. Further support has been provided by the DGICYT under Project No. PB91-0221. The hospitality of the Department of Physics, University of Miami, is also appreciated.

APPENDIX

In this Appendix we will obtain the term $\hat{E}^{[0]}$ to have the complete leading-order solution for the problem of finite electrical resistivity. In order to avoid further complication of the notation we will use Γ instead of $\Gamma^{[0]}$, because the next contribution $\Gamma^{[2]}$ will not appear to the order in the asymptotic expansion in $1/R$ we are working at. Taking into account the expansions of Eq. (70), to order $R^{1/2}$, Eq. (62) yields

$$2\frac{dq}{dx}\left[2\left[\frac{dq}{dx}\right]^2\frac{F}{\Gamma}-x\left[1-\frac{x}{2}\right]\right]\frac{d\hat{b}_3^{[0]}}{dx}+\left\{\left[\left[\frac{dq}{dx}\right]^2\frac{6F}{\Gamma}-x\left[1-\frac{x}{2}\right]\right]\frac{d^2q}{dx^2}+\frac{dq}{dx}(-1+x)+\left[\frac{dq}{dx}\right]^3\frac{2}{\Gamma}\frac{dF}{dx}\right\}\hat{b}_3^{[0]}=0\ , \tag{A1}$$

and exactly the same equation for $\hat{b}_4^{[0]}$. Taking into account Eq. (69), we rewrite Eq. (A1) as

$$\frac{1}{\hat{b}_3^{[0]}}\frac{d\hat{b}_3^{[0]}}{dx}=-\frac{5}{2}\frac{d^2q/dx^2}{dq/dx}+\frac{1-x}{2x(1-x/2)}-\frac{dF/dx}{F},$$

which may be easily integrated, resulting in

$$\hat{b}_3^{[0]}=\hat{b}_4^{[0]}=\left\{\Gamma^5 x\left[1-\frac{x}{2}\right]\left[(1-x)^2+x\left[1-\frac{x}{2}\right]\right]\right\}^{-1/4}.$$

Using this result and Eq. (69) we get $\hat{b}_3$ and $\hat{b}_4$ to leading order in $1/R$ as

$$\hat{b}_3=\left\{\Gamma^5 x\left[1-\frac{x}{2}\right]\left[(1-x)^2+x\left[1-\frac{x}{2}\right]\right]\right\}^{-1/4}\exp\left\{R^{1/2}\Gamma^{1/2}\int_0^x\left[\frac{(1-t)^2+t(1-t/2)}{t(1-t/2)}\right]^{1/2}dt\right\}, \tag{A2}$$

and

$$\hat{b}_4=\left\{\Gamma^5 x\left[1-\frac{x}{2}\right]\left[(1-x)^2+x\left[1-\frac{x}{2}\right]\right]\right\}^{-1/4}\exp\left\{-R^{1/2}\Gamma^{1/2}\int_0^x\left[\frac{(1-t)^2+t(1-t/2)}{t(1-t/2)}\right]^{1/2}dt\right\}. \tag{A3}$$

To leading order their values at $x=1$ are

$$\hat{b}_3(x=1)=\left[\frac{4}{\Gamma^5}\right]^{1/4}\exp\left\{R^{1/2}\Gamma^{1/2}\int_0^1\left[\frac{(1-t)^2+t(1-t/2)}{t(1-t/2)}\right]^{1/2}dt\right\} \tag{A4}$$

and

$$\hat{b}_4(x=1)=\left[\frac{4}{\Gamma^5}\right]^{1/4}\exp\left\{-R^{1/2}\Gamma^{1/2}\int_0^1\left[\frac{(1-t)^2+t(1-t/2)}{t(1-t/2)}\right]^{1/2}dt\right\}, \tag{A5}$$

while their second derivatives are

$$\frac{d^2\hat{b}_3}{dx^2}=R\Gamma\hat{b}_3 \quad\text{and}\quad \frac{d^2\hat{b}_4}{dx^2}=R\Gamma\hat{b}_4 \quad\text{at } x=1\ . \tag{A6}$$

The boundary condition of Eq. (63) at the plasma-solid surface requires the vanishing of the second derivative of $\hat{b}$ at $x=1$. To leading order, this condition leads to the relation given by Eq. (74). It is easy to see, using Eq. (75) at $x=1$, together with the boundary condition Eq. (77), that $d^2\hat{A}^{[0]}/dx^2$ vanishes at $x=1$. Also, from Eq. (118), the second derivative of $\hat{A}^{[1]}$ vanishes at $x=1$. Therefore, from the expansion of Eq. (73), the nonvanishing leading-order contribution to the second derivative of $\hat{A}$ at the plasma-solid surface is given by

$$\left[\frac{d^2\hat{A}}{dx^2}\right]^{[\mathrm{LO}]}=\frac{1}{R}\frac{d^2\hat{A}^{[2]}}{dx^2}\quad\text{at } x=1\ .$$

Using Eq. (119) at $x=1$, together with Eqs. (120) and (77), and noting that

$$\frac{dF}{dx}(x=1)=0\ ,$$

we get

$$\left[\frac{d^2\hat{A}}{dx^2}\right]^{[\mathrm{LO}]}=\frac{1}{R\Gamma^{[0]}}\frac{d^4\hat{A}^{[0]}}{dx^4}\quad\text{at } x=1\ .$$

Differentiating Eq. (75) twice and applying the results at $x=1$, the fourth derivative of $\hat{A}^{[0]}$ at the plasma-solid surface is obtained. Thus, from the previous relation, we get

$$\left[\frac{d^2\hat{A}}{dx^2}\right]^{[\mathrm{LO}]}=\frac{4k^2}{R(\Gamma^{[0]})^3}\frac{d\hat{A}^{[0]}}{dx}\quad\text{at } x=1\ . \tag{A7}$$

Therefore, from Eq. (74), using Eq. (A6), we get

$$\frac{1}{R^2}\frac{4k^2}{(\Gamma^{[0]})^3}\frac{d\hat{A}^{[0]}}{dx}+c_3\Gamma\hat{b}_3+c_4\Gamma\hat{b}_4=0\quad\text{at } x=1\ . \tag{A8}$$

This shows that the leading-order term of the exponential contribution, given by $\hat{E}^{[0]}$, is in fact of order R^{-2} with respect to the leading-order term of the algebraic contribution, $\hat{A}^{[0]}$. We still have to impose that the solution satisfies the boundary condition at the plasma-vacuum interface $x=0$. We will see below that this makes $c_4=0$. Therefore Eq. (A8), together with Eqs. (84) and (85), determines c_3 in terms of c_1. In order to apply the boundary conditions at $x=0$ we must realize that the expansions of Eq. (70) are not valid near $x=0$. The expansions are valid only as long as the exponent in $\hat{b}_3$ and $\hat{b}_4$ is large, and lose their validity when the exponent becomes of order unity. For small values of x, we may take the exponent

$$R^{1/2}q(x)\approx 2(\Gamma R x)^{1/2}\quad\text{for } x\ll 1\ . \tag{A9}$$

Therefore, when $x\sim R^{-1}$, the expansions are no longer valid. Then, for the region near $x=0$, we define the stretched variable

$$\eta=Rx\ . \tag{A10}$$

In terms of this variable η, for small values of x, but η of order unity (that is, for $x\sim R^{-1}$), Eq. (62), to leading order (order R), reduces to

$$\eta^2\frac{d^4\hat{b}}{d\eta^4}+4\eta\frac{d^3\hat{b}}{d\eta^3}+(2-\eta\Gamma)\frac{d^2\hat{b}}{d\eta^2}-\Gamma\frac{d\hat{b}}{d\eta}=0\ ,$$

which may be rewritten as

$$\frac{d}{d\eta}\left[\frac{d}{d\eta}\left[\eta^2\frac{d^2\hat{b}}{d\eta^2}\right]\right]-\Gamma\frac{d}{d\eta}\left[\eta\frac{d\hat{b}}{d\eta}\right]=0\ .$$

A first integration of this equation provides

$$\frac{d}{d\eta}\left[\eta^2\frac{d^2\hat{b}}{d\eta^2}\right]-\Gamma\eta\frac{d\hat{b}}{d\eta}=\text{const}\ . \tag{A11}$$

The general solution of this last equation may be written as the sum of the general solution of the homogeneous equation plus any particular solution of the inhomogeneous equation. A particular solution is

$$\hat{b}_{\text{part}}=-\frac{\text{const}}{\Gamma}\ln\eta\ . \tag{A12}$$

On the other hand, by defining

$$\bar{\eta}=2\Gamma^{1/2}\eta^{1/2}=2(\Gamma Rx)^{1/2}\ , \tag{A13}$$

the homogeneous part of Eq. (A11) transforms to

$$\bar{\eta}^2\frac{d^2}{d\bar{\eta}^2}\left[\frac{d\hat{b}}{d\bar{\eta}}\right]+\bar{\eta}\frac{d}{d\bar{\eta}}\left[\frac{d\hat{b}}{d\bar{\eta}}\right]-(1+\bar{\eta}^2)\frac{d\hat{b}}{d\bar{\eta}}=0\ . \tag{A14}$$

The solution for $d\hat{b}/d\bar{\eta}$ is

$$\frac{d\hat{b}}{d\bar{\eta}}=d_3I_1(\bar{\eta})-d_4K_1(\bar{\eta})\ ,$$

where d_3, d_4 are two constants. The sign in front of d_4 has been chosen negative for convenience, and I_1 and K_1 are the modified Bessel functions [A16]. Then, integrating the last equation and adding the particular solution Eq. (A12), the general solution of Eq. (A11) may be written as

$$\hat{b}=d_1+d_2\ln\eta+d_3I_0(2\Gamma^{1/2}\eta^{1/2})+d_4K_0(2\Gamma^{1/2}\eta^{1/2})\ . \tag{A15}$$

For very small values of the argument, I_0 tends to unity, whereas $K_0(\bar{\eta})\to-[\ln(\bar{\eta}/2)+\gamma]$, with γ the Euler constant. Thus, as η goes to zero

$$\hat{b}(\eta\to 0)\approx d_1+d_2\ln\eta+d_3-d_4[\ln(\Gamma^{1/2}\eta^{1/2})+\gamma]\ . \tag{A16}$$

The boundary condition of Eq. (64) at the plasma-vacuum interface imposes that $\hat{b}$ remains finite there. Therefore $d_4=2d_2$.

On the other hand,

$$\frac{d^2I_0}{d\eta^2}=\frac{\Gamma}{\eta}\left[\frac{d^2I_0}{d\bar{\eta}^2}-\frac{1}{\bar{\eta}}\frac{dI_0}{d\bar{\eta}}\right]=\frac{\Gamma}{\eta}\left[\frac{dI_1}{d\bar{\eta}}-\frac{I_1}{\bar{\eta}}\right]=\frac{\Gamma}{\eta}I_2(2\Gamma^{1/2}\eta^{1/2})\ ,$$

and

$$\frac{d^2K_0}{d\eta^2}=\frac{\Gamma}{\eta}\left[\frac{d^2K_0}{d\bar{\eta}^2}-\frac{1}{\bar{\eta}}\frac{dK_0}{d\bar{\eta}}\right]=-\frac{\Gamma}{\eta}\left[\frac{dK_1}{d\bar{\eta}}-\frac{K_1}{\bar{\eta}}\right]=\frac{\Gamma}{\eta}K_2(2\Gamma^{1/2}\eta^{1/2})\ .$$

Thus

$$\frac{d^2\hat{b}}{d\eta^2}=-\frac{d_2}{\eta^2}+d_3\frac{\Gamma}{\eta}I_2(2\Gamma^{1/2}\eta^{1/2})+d_4\frac{\Gamma}{\eta}K_2(2\Gamma^{1/2}\eta^{1/2})\ .$$

For small values of the argument $I_2(\bar{\eta})$ goes as $\bar{\eta}^2/8$, and $K_2(\bar{\eta}\to 0)\approx 2/\bar{\eta}^2-\frac{1}{2}$. Therefore

$$\frac{d^2\hat{b}}{d\eta^2}\approx-\frac{d_2}{\eta^2}+d_3\frac{\Gamma^2}{2}+d_4\left[\frac{1}{2\eta^2}-\frac{\Gamma}{2\eta}\right]\quad\text{when }\eta\to 0\ .$$

The boundary condition Eq. (64) requires that $\eta d^2\hat{b}/d\eta^2$ should vanish as η goes to zero. Thus, from the previous relation, and using the former results that $d_4=2d_2$, we obtain

$$d_2=d_4=0\ .$$

Therefore, from Eq. (A15) we may write, near the point $x=0$,

$$\hat{b}=d_1+d_3I_0(2\Gamma^{1/2}R^{1/2}x^{1/2})\ ,$$

which for large values of the argument of the function I_0 behaves as

$$\hat{b}\approx d_1+d_3\frac{e^{2(\Gamma Rx)^{1/2}}}{2\pi^{1/2}(\Gamma Rx)^{1/4}}\quad\text{for }2(\Gamma Rx)^{1/2}\gg 1\ ,\quad\text{but }x\ll 1\ . \tag{A17}$$

On the other hand, from Eq. (71), taking into account Eqs. (A2) and (A3), we obtain that for small values of x,

$$\hat{b}(x\ll 1)\approx A^{[0]}(x=0)+c_3\frac{e^{2(\Gamma Rx)^{1/2}}}{(\Gamma^5x)^{1/4}}+c_4\frac{e^{-2(\Gamma Rx)^{1/2}}}{(\Gamma^5x)^{1/4}}\ . \tag{A18}$$

The matching between Eqs. (A17) and (A18) provides the relations

$$d_1=\hat{A}^{[0]}(x=0)\ ,\quad d_3=\frac{2\pi^{1/2}R^{1/4}}{\Gamma}c_3\ ,\quad\text{and }c_4=0\ . \tag{A19}$$

Equation (A19) determines the values of the two constants d_1 and d_3 and imposes the vanishing of the coefficient c_4. We substitute Eqs. (A4) and (A19) into Eq. (A8) to get c_3 and d_3 in terms of $d\hat{A}^{[0]}/dx$ (at $x=1$). The result is

$$c_3 = d_3 \frac{\Gamma}{2\pi^{1/2} R^{1/4}}$$
$$= -\frac{1}{R^2} \frac{4k^2}{\Gamma^3} \left[\frac{\Gamma}{4}\right]^{1/4} \exp\left\{-R^{1/2}\Gamma^{1/2} \int_0^1 \left[\frac{(1-t)^2 + t(1-t/2)}{t(1-t/2)}\right]^{1/2} dt\right\} \left[\frac{d\hat{A}^{[0]}}{dx}\bigg|_{x=1}\right] . \tag{A20}$$

All the different coefficients become specified once c_1 is given. Since the equation was linear a coefficient always remains unspecified to be determined by the initial amplitude of the perturbations.

In summary, we have obtained that for large but finite values of the parameter R, the solution of Eq. (62), to leading order in an expansion in the smallness parameter $1/R$, may be written as

$$\hat{b} = \hat{A}^{[0]} + \hat{E}^{[0]} = \hat{A}^{[0]} + c_3 \hat{b}_3^{[0]} \quad \text{for } x \text{ of order unity} . \tag{A21}$$

We emphasize that $\hat{A}^{[0]}$ is the part of the solution that is algebraic in R, defined in Eq. (73). $\hat{A}^{[0]}$ was obtained in Sec. VII, leading to a determination of $\Gamma^{[0]}$ vs k.

As a consequence, when R is large but finite, Eq. (A21) indicates that for values of x of order unity the magnetic-field solution is given by the solution $\hat{A}^{[0]}$ for $R=\infty$, plus the term $\hat{E}^{[0]}$, which, using Eqs. (A2) and (A20), gives

$$\hat{E}^{[0]} = c_3 \hat{b}_3 = -\frac{1}{R^2} \frac{4k^2}{\Gamma^4} \left[\frac{d\hat{A}^{[0]}}{dx}\bigg|_{x=1}\right] \left\{4x\left[1-\frac{x}{2}\right]\left[(1-x)^2 + x\left[1-\frac{x}{2}\right]\right]\right\}^{-1/4}$$
$$\times \exp\left\{-R^{1/2}\Gamma^{1/2} \int_x^1 \left[\frac{(1-t)^2 + t(1-t/2)}{t(1-t/2)}\right]^{1/2} dt\right\} . \tag{A22}$$

Therefore the term $\hat{E}^{[0]}$ is small, not only because it contains a factor R^{-2}, but because the integrand in the exponential is negative, making this term exponentially small.

For small values of x, instead of the relation of Eq. (A21), we should write

$$\hat{b} = \hat{A}^{[0]} + \hat{E}^{[0]} = \hat{A}^{[0]} + d_3 I_0(2[\Gamma R x]^{1/2}) \quad \text{for } Rx \text{ of order unity} . \tag{A23}$$

Using Eq. (A20), the contribution $d_3 I_0$ is

$$d_3 I_0(2[\Gamma R x]^{1/2}) = -\frac{1}{R^{7/4}} \frac{4k^2(2\pi)^{1/2}}{\Gamma^{15/4}} \exp\left\{-R^{1/2}\Gamma^{1/2} \int_0^1 \left[\frac{(1-t)^2 + t(1-t/2)}{t(1-t/2)}\right]^{1/2} dt\right\}$$
$$\times \left[\frac{d\hat{A}^{[0]}}{dx}\bigg|_{x=1}\right] I_0(2[\Gamma R x]^{1/2}) . \tag{A24}$$

The contribution of Eq. (A24) introduced by R finite is again exponentially small for small values of x (such that Rx is small, or of order unity, at most.)

Therefore, from Eqs. (A22) and Eq. (A24), we obtain that for the whole range of values of x, the magnetic-field perturbation $\hat{b}$, for large values of R, is almost equal to the solution $\hat{A}^{[0]}$, corresponding to $R=\infty$.

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